Double Field Theory and Closed String T-Duality

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Abstract. Double Field Theory is a formalisation of string theory in which T-duality, a uniquely stringy phenomenon, is promoted to a manifest symmetry. The theory of closed strings on toroidal backgrounds is reviewed, leading to the notion of dual coordinates. The Double Field Theory, invariant under the $O(d, d, \mathbb{R})$ duality group, is then constructed and its local symmetries discussed. An $O(d, d, \mathbb{R})$ invariant action is then presented which, upon imposing the section condition, is shown to reduce to the NS-NS sector supergravity action.

Contents

Declaration

I hereby certify that this project report, which is approximately eight and a half thousand words in length, has been written by me at the School of Physics and Astronomy, Queen Mary University of London, that all material in this dissertation which is not my own work has been properly acknowledged, and that it has not been submitted in any previous application for a degree.

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1. Introduction

1.1. General Introduction

A duality between theories hints at a potential unification between them, and at the possibility that the theories in question are special cases of a more general one. Dualities are relations through which physical theories are equated. It is this concept with which this report will be concerned. More specifically, we will consider the T-duality relating different string theories. T-duality is a hidden symmetry arising in theories with backgrounds constrained by isometry. By doubling the dimensions of the space-time, a new theory, dubbed *Double Field Theory*, can be constructed. This approach promotes T-duality to a manifest symmetry. In doing so, the T-dual theories become special cases of the Double Field Theory.

In string theory, the bosonic and fermionic state spaces result from choosing either Neveu-Schwarz (NS) or Ramond (R) boundary conditions. There are four closed string sectors, each constructing a spectrum of states from one left-moving sector and one right-moving sector.[1] The resulting four sectors are:

NS-NS NS-R R-NS R-R

In this report we will focus on the NS-NS sector of closed bosonic strings. The R-R sector, and the fermionic NS-R and R-NS sectors, will not be considered.

We will begin by reviewing the T-duality of the closed bosonic string theory. After this the $O(d,d,\mathbb{R})$ duality group, from which the Double Field Theory is constructed, is discussed. With this Double Field Theory in place, one can then begin to explore it's properties. The local symmetries of the theory are investigated, and are related to the local symmetries of the original theory by a reduction similar to that of Kaluza-Klein theory. After examining these local symmetries, and their gauge algebra, an action to describe the Double Field Theory is presented. It is then shown that, upon imposition of a suitable condition, this action can be reduced to the NS-NS sector supergravity action.

1.2. Closed Bosonic Strings

In this section the basics of closed bosonic string theory are discussed. This is not a full review, instead it is a minimal presentation of the tools required for the remainder of the report. We will be considering the NS-NS sector; the massless fields of such a theory are:

- A symmetric field G. This field is the metric of the background space-time.
- The antisymmetric Kalb-Ramond gauge field B.
- A scalar field Φ, known as the dilaton.

Given these massless excitations of the string, one can write an action for the theory:

$$S = \frac{-1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left(\sqrt{|h|} h^{\mu\nu} G_{ij} (\partial_\mu X^i)(\partial_\nu X^j) + \epsilon^{\mu\nu} B_{ij} (\partial_\mu X^i)(\partial_\nu X^j) + \sqrt{|h|} \alpha' R^{(2)} \Phi \right) \tag{1.1}$$

This action is essentially a Polyakov action, with the inclusion of terms coupling the B field, and the dilaton, to the string.

As the string propagates it sweeps out a two-dimensional surface. This surface is known as the string worldsheet, and is the string theory analogue of a point particle's worldline. This surface is parametrised by the worldsheet coordinates τ and σ. τ is a timelike coordinate with the range $\tau \in [-\infty, \infty]$. σ corresponds to the spacelike coordinate along the length of the string. Since the string is closed we find that σ is periodic in 2π, with the range $\sigma \in [0, 2\pi)$. The objects h and $R^{(2)}$ in the above action are the metric on the worldsheet, and the worldsheet's scalar curvature, respectively.

The string is a one-dimensional object embedded in a background space-time known as the target space.[1] The string's position in this target space is then given by the string coordinates $X^i(\tau, \sigma)$. It is necessary to specify which point on the string we are referring to. As such, the string coordinates are in fact functions of the worldsheet coordinates. We will encounter derivatives of the string coordinates with respect to τ and σ. To aid readability, the following conventions are used:

$$\dot{X}^i := \partial_\tau X^i \tag{1.2}$$

$$X'^i := \partial_\sigma X^i \tag{1.3}$$

The integration measure in the action, (1.1), is a shorthand notation with $\mathrm{d}^2\sigma := \mathrm{d}\tau\, \mathrm{d}\sigma$. That is to say, the action is an integration over both of the worldsheet coordinates.

Each string coordinate can be decomposed into a left-moving part and a right-moving part as follows: $X^i(\tau, \sigma) = X^i_L(\tau + \sigma) + X^i_R(\tau - \sigma)$. As we will see in 2.3, these left-moving and right-moving oscillators can be expanded in terms of the modes α_n^i and $\bar{\alpha}_n^i$ respectively. After quantising the theory these modes correspond to creation and annihilation operators, of the type seen in the quantum harmonic oscillator.

The constant α' appearing in (1.1) is the *slope parameter*. It has dimensions of length-squared, and is related to the string tension by:

$$T = \frac{1}{2\pi\alpha'} \tag{1.4}$$

1.3. Conventions

Unless stated otherwise, the following conventions will be used in this report:

- We will work in natural units; i.e., $\hbar = c = 1$.
- Greek characters (μ, ν, $\ldots$) will be used to label indices corresponding to the worldsheet coordinates, τ and σ.
- Lowercase latin indices (i, j, $\ldots$) will be run of d coordinates. In the case of the Double Field Theory, these lowercase labels correspond to a d-dimensional subset of the total $2d$ coordinates.
- Uppercase latin characters (M, N, $\ldots$) will be used to denote $O(d, d)$ indices. Such indices correspond to the $2d$-dimensional space-time.

2. T-Duality

2.1. The Winding Modes of Closed Strings

The T-duality of closed strings is a phenomenon arising in theories with toroidal backgrounds; i.e, geometries with periodic isometry. In this section we will examine a simple example of such a geometry, namely a two-dimensional cylinder. The coordinates x and y will be used to describe this surface, with the x denoting the closed dimension, and y the direction along the length of the cylinder. One can then describe the periodicity of x by the identification:[1]

$$(x, y) \sim (x + 2\pi R, y) \tag{2.1}$$

Where, R is the radius of the circular dimension. Figure 2.1 provides a pictorial representation of such a geometry.

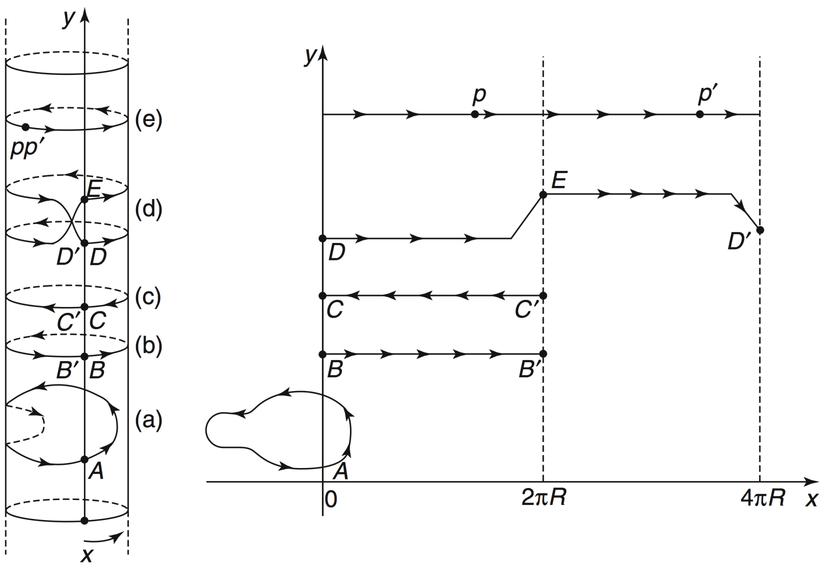

Figure 2.1.: Left: a collection of closed strings living on the surface of a two-dimensional cylinder. Right: the same set of strings represented on the covering space of the cylinder. Strings with nontrivial winding numbers appear in the covering space as open strings.[1]

The strings in this background exhibit a new behaviour, they can wrap around the closed dimension. Since these strings are closed they cannot be unwound from the cylinder. Additionally, we see that the circumference of the circular dimension defines a minimum size for strings wrapped around it. We can define a *winding number*, $m \in \mathbb{Z}$, to count the number of times a string is wound around the cylinder. The aforementioned figure displays five examples of how strings could be wrapped around the cylinder, with arrows denoting the positive σ direction along each string. We will use the convention that m is positive when the positive σ direction is the same as the positive x direction, and m is negative otherwise. Calculating the winding number for the pictured strings is simply a case of counting how many times they wrap the cylinder, thus:

(a) $m = 0$ (b) $m = 1$ (c) $m = -1$ (d) $m = 2$ (e) $m = 2$

Due to the geometric nature of this winding procedure, we say that m is a *topological* quantum number.[2] One can use m to define an object, the *winding*, with dimensions of momentum:

$$\omega := \frac{mR}{\alpha'} \tag{2.2}$$

The reasoning for defining such a quantity, with dimensions of momentum, will become clear when we examine the oscillators in 2.3.

Recall that the string coordinate $X(\tau, \sigma)$ is periodic under $\sigma \to \sigma + 2\pi$. However, each time we complete one of these periodicities we move around the circular dimension m times. Such a translation is of the following form:

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + 2\pi m R \tag{2.3}$$

$$= X(\tau, \sigma) + 2\pi\alpha'\omega \tag{2.4}$$

It is through this periodicity condition that the winding will enter the theory.

2.2. Momentum Quantisation on Closed Dimensions

The simple system from 2.1 provides the notion of a quantised winding number. However, the introduction of a closed dimension also has the consequence of quantising the momentum.[1] To see this we will consider the simple example of a quantum mechanical position state $|x, y\rangle$. Given such a state, one can define an operator corresponding to a translation in the x direction:

$$\hat{T}_x(a) |x, y\rangle := |x + a, y\rangle \tag{2.5}$$

Such a translation operator can be represented in terms of the x-direction momentum p_x:

$$\hat{T}_x(a) = e^{-iap_x} \tag{2.6}$$

One can then consider what happens when we translate $2\pi R$ along the x-direction. Using the identification (2.1), we see that such a translation becomes:

$$\hat{T}_x(2\pi R) |x, y\rangle = |x + 2\pi R, y\rangle \tag{2.7}$$
$$= |x, y\rangle \tag{2.8}$$

That is to say, the operation of translating $x \to x + 2\pi R$ is the identity operation. Using the representation in (2.6), we see that:

$$e^{-i2\pi Rp_x} |x, y\rangle = |x, y\rangle \tag{2.9}$$

Hence, we find a condition on the x-direction momentum:

$$\cos(2\pi Rp_x) - i\sin(2\pi Rp_x) = 1 \tag{2.10}$$

Clearly the sine function is zero, and the cosine unity. Since cosine is periodic, we introduce the quantum number $n \in \mathbb{Z}$ such that:

$$\cos(2\pi Rp_x) = \cos(2\pi n) \tag{2.11}$$

This then gives an equation for the momentum that is quantised:

$$p_x = \frac{n}{R} \tag{2.12}$$

Whilst basic, this result is important. In general, for each closed dimension we have two discrete quantities: the momentum and the winding.

2.3. A Glimpse of a Hidden Symmetry

We will now consider the traditional notion of the closed bosonic string in 26-dimensions, with the additional requirement that the 25^{th} spacial dimension be closed. That is, given the light-cone string coordinates

$$X(\sigma, \tau) = (X^+, X^-, X^2, X^3, \ldots, X^{24}, X^{25}) \tag{2.13}$$

the coordinate X^{25} describes the closed dimension. In this section we will use the convention that latin indices label the non-compact spacial dimensions; i.e, $i \in \{2, 3, \ldots, 24\}$.

We can write X^{25} in terms of left and right moving oscillators:

$$X^{25}(\sigma, \tau) = X_L^{25}(\tau + \sigma) + X_R^{25}(\tau - \sigma) \tag{2.14}$$

As expected, the expansions of X_L^{25} and X_R^{25} in terms of left moving and right moving modes α_n^{25} and $\bar{\alpha}_n^{25}$ respectively are:

$$X_L^{25}(\tau + \sigma) = \frac{x_0^L}{2} + \sqrt{\frac{\alpha'}{2}}\alpha_0^{25}(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^{25}}{n} e^{-in(\tau+\sigma)} \tag{2.15}$$

$$X_R^{25}(\tau - \sigma) = \frac{x_0^R}{2} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^{25}(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\bar{\alpha}_n^{25}}{n} e^{-in(\tau-\sigma)} \tag{2.16}$$

Imposing the new periodicity condition, (2.4), one finds that:

$$X_L^{25}(\tau + \sigma + 2\pi) - X_L^{25}(\tau + \sigma) = X_R^{25}(\tau - \sigma) - X_R^{25}(\tau - \sigma - 2\pi) + 2\pi\alpha'\omega \tag{2.17}$$

Substituting the expanded forms of X_L^{25} and X_R^{25} into this expansion then yields:

$$\alpha_0^{25} - \bar{\alpha}_0^{25} = \sqrt{2\alpha'}\omega \tag{2.18}$$

Interestingly, the introduction of a closed dimension has caused this result to be non-zero, and proportional to the winding. One can also calculate the momentum p along the closed dimension:

$$p = \frac{1}{2\pi\alpha'} \int_0^{2\pi} \mathrm{d}\sigma \dot{X}^{25} \tag{2.19}$$

$$= \frac{1}{2\pi} \frac{1}{\sqrt{2\alpha'}} \int_0^{2\pi} \mathrm{d}\sigma \left(\alpha_0^{25} + \bar{\alpha}_0^{25} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \alpha_n^{25} e^{-in(\tau+\sigma)} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \bar{\alpha}_m^{25} e^{-im(\tau-\sigma)} \right) \tag{2.20}$$

$$= \frac{1}{\sqrt{2\alpha'}} (\alpha_0^{25} + \bar{\alpha}_0^{25}) \tag{2.21}$$

Hence, we can write the zero modes in terms of the momentum and winding:

$$\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}(p + \omega) \tag{2.22}$$

$$\bar{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}(p - \omega) \tag{2.23}$$

In order to see the T-duality, we will calculate the Hamiltonian of the theory:

$$\mathscr{H} = L_0^{\perp} + \bar{L}_0^{\perp} - 2 \tag{2.24}$$

Where the Virasoro operators are given by:

$$L_0^{\perp} = \frac{1}{2}\sum_i \alpha_0^i \alpha_0^i + \frac{1}{2}\alpha_0^{25}\alpha_0^{25} + N^{\perp} \tag{3.26}$$

$$\bar{L}_0^{\perp} = \frac{1}{2}\sum_i \bar{\alpha}_0^i \bar{\alpha}_0^i + \frac{1}{2}\bar{\alpha}_0^{25}\bar{\alpha}_0^{25} + \bar{N}^{\perp} \tag{3.27}$$

Using these expressions the Hamiltonian can be rewritten as:

$$\mathscr{H} = \frac{1}{2}\left(\sum_i \alpha_0^i \alpha_0^i + \sum_j \bar{\alpha}_0^j \bar{\alpha}_0^j + \alpha_0^{25}\alpha_0^{25} + \bar{\alpha}_0^{25}\bar{\alpha}_0^{25}\right) + N^{\perp} + \bar{N}^{\perp} - 2 \tag{2.27}$$

Writing α_0^{25} and $\bar{\alpha}_0^{25}$ in terms of p and w, and using the standard result for the open dimensions, gives:

$$\mathscr{H} = \frac{\alpha'}{2}\left(\sum_i p^i p^i + p^2 + w^2\right) + N^{\perp} + \bar{N}^{\perp} - 2 \tag{2.28}$$

Finally, this can be rewritten in terms of the momentum and winding quanta n and m:

$$\mathscr{H} = \frac{\alpha'}{2}\left(\sum_i p^i p^i + \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2}\right) + N^{\perp} + \bar{N}^{\perp} - 2 \tag{2.29}$$

This Hamiltonian is invariant under the following set of transformations:

$$\left\{\frac{R}{\sqrt{\alpha'}} \to \frac{\sqrt{\alpha'}}{R} \qquad n \to m \qquad m \to n\right\} \tag{2.30}$$

This set of mappings is the hidden symmetry known as T-duality. We see that the Hamiltonian of a theory with a closed dimension of radius R is identified with that of a theory with a closed dimension of radius α'/R, provided that the momentum and winding quantum numbers are interchanged. That is to say that in the dual theory, n becomes the quantum number of the winding and m the quantum number of the momentum. This symmetry of the Hamiltonian hints at an equivalence between the two theories. In the next section, we will see how the T-duality relates physically indistinguishable theories.

2.4. The T-Duality Transformation

We will now consider the case of the closed string in a d-dimensional space-time in the presence of two background fields: a metric G and an antisymmetric Kalb-Ramond field B. We then require that at least one of the dimensions exhibits a continuous abelian isometry.[3] For example: a closed, circular, dimension has continuous isometry described by the circle group U(1). We will see that the T-duality, hinted at in the previous section, is a symmetry of such a theory. An action for this closed string theory is given by:

$$S = \frac{-1}{4\pi}\int \mathrm{d}^2\sigma \left(\sqrt{|h|}h^{\mu\nu}G_{ij}(\partial_\mu X^i)(\partial_\nu X^j) + \epsilon^{\mu\nu}B_{ij}(\partial_\mu X^i)(\partial_\nu X^j) + \sqrt{|h|}R^{(2)}\Phi \right) \tag{2.31}$$

Here $R^{(2)}$ is the scalar curvature of the worldsheet, and Φ is a scalar field known as the dilaton. This dilaton term is not necessary for the T-duality to be observed; however, we will later note how it transforms under the duality. The G and B fields are chosen to be dimensionless such that we need not include factors of α' in the action. The coordinate describing the dimension with isometry will be denoted X^θ. After expanding the summations to write the X^θ term explicitly, the action becomes:

$$\begin{aligned} S = \frac{-1}{4\pi}\int \mathrm{d}^2\sigma \Big(&\sqrt{|h|}h^{\mu\nu}G_{pq}(\partial_\mu X^p)(\partial_\nu X^q) + \epsilon^{\mu\nu}B_{pq}(\partial_\mu X^p)(\partial_\nu X^q) \\ &+ \sqrt{|h|}h^{\mu\nu}\left[G_{\theta\theta}(\partial_\mu X^\theta)(\partial_\nu X^\theta) + 2G_{\theta q}(\partial_\mu X^\theta)(\partial_\nu X^q)\right] \\ &+ 2\epsilon^{\mu\nu}B_{\theta q}(\partial_\mu X^\theta)(\partial_\nu X^q) + \sqrt{|h|}R^{(2)}\Phi \Big) \end{aligned} \tag{2.32}$$

Previously i and j ran over all d coordinates, here p and q correspond all coordinates except θ. Given this action, one can now *gauge the isometry* by introducing a gauge field A^θ coupled to a Lagrange multiplier $\tilde{X}^\theta$. The isometry of X^θ admits a Killing vector ∂_θ, such that the background fields are locally independent of X^θ. We can then write a new action by substituting ${A^\theta}_\mu$ in place of $\partial_\mu X^\theta$.[4] This method brings the isometry constraint into the action directly, yielding:[5]

$$\begin{aligned} S_{\text{gauged}} = \frac{-1}{4\pi}\int \mathrm{d}^2\sigma \Big(&\sqrt{|h|}h^{\mu\nu}G_{pq}(\partial_\mu X^p)(\partial_\nu X^q) + \epsilon^{\mu\nu}B_{pq}(\partial_\mu X^p)(\partial_\nu X^q) \\ &+ \sqrt{|h|}h^{\mu\nu}\left[G_{\theta\theta}{A^\theta}_\mu {A^\theta}_\nu + 2G_{\theta q}{A^\theta}_\mu(\partial_\nu X^q)\right] \\ &+ 2\epsilon^{\mu\nu}B_{\theta q}{A^\theta}_\mu(\partial_\nu X^q)\big] + 2\tilde{X}^\theta\epsilon^{\mu\nu}(\partial_\mu {A^\theta}_\nu) + \sqrt{|h|}R^{(2)}\Phi \Big) \end{aligned} \tag{2.33}$$

Let's take a moment to inspect this action. The equation of motion for the Lagrange multiplier gives:

$$\epsilon^{\mu\nu}(\partial_\mu A^\theta{}_\nu) = 0 \tag{2.34}$$

Implementing this equation causes the derivative of A^θ to vanish from the action. Given this constraint, the equation of motion for the gauge field is simply:

$$\sqrt{|h|}h^{\mu\nu}\left[G_{\theta\theta}A^\theta{}_\nu + G_{\theta q}(\partial_\nu X^q)\right] + \epsilon^{\mu\nu}B_{\theta q}(\partial_\nu X^q) = 0 \tag{2.35}$$

One can then use this result to simplify the action. In fact, one finds that this gives precisely the action (2.32). Thus, we see that A^θ is pure gauge; that is, the inclusion of A^θ and $\tilde{X}^\theta$ amounts to introducing trivial terms to the action.[6] Hence, any new action that we find must be equivalent to the original action that we started with.

Instead of immediately applying the pure gauge condition (2.34), we will consider the unconstrained equation of motion for the gauge field. First, it is useful to integrate by parts the derivative of A^θ:

$$\int \mathrm{d}^2\sigma \tilde{X}^\theta \epsilon^{\mu\nu}(\partial_\mu A^\theta{}_\nu) = -\int \mathrm{d}^2\sigma \epsilon^{\mu\nu}(\partial_\mu \tilde{X}^\theta)A^\theta{}_\nu \tag{2.36}$$

Here we have ignored the boundary term; we assume that the gauge field A^θ vanishes at the boundaries of the space-time. Given this, the equation of motion for $A^\theta{}_\nu$ is:

$$\sqrt{|h|}h^{\mu\nu}\left[G_{\theta\theta}A^\theta{}_\mu + G_{\theta q}(\partial_\mu X^q)\right] + \epsilon^{\mu\nu}\left[B_{\theta q}(\partial_\mu X^q) - (\partial_\mu \tilde{X}^\theta)\right] = 0 \tag{2.37}$$

This can then be rearranged to give an equation for the gauge field:

$$A^\theta{}_\sigma = \frac{1}{\sqrt{|h|}}\frac{1}{G_{\theta\theta}}h_{\sigma\nu}\epsilon^{\mu\nu}\left[(\partial_\mu \tilde{X}_\theta) - B_{\theta q}(\partial_\mu X^q)\right] - \frac{G_{\theta q}}{G_{\theta\theta}}(\partial_\sigma X^q) \tag{2.38}$$

This can then be substituted back into the action (2.33), giving an equation no longer written in terms of the gauge field. After some simplification this becomes:

$$\begin{aligned}\tilde{S} = \frac{-1}{4\pi}\int \mathrm{d}^2\sigma\Big(&\sqrt{|h|}h^{\mu\nu}(G_{pq} - G_{\theta p}G_{\theta q}G_{\theta\theta}^{-1} + B_{\theta p}B_{\theta q}G_{\theta\theta}^{-1})(\partial_\mu X^p)(\partial_\nu X^q)\\ &+ \epsilon^{\mu\nu}(B_{pq} + G_{\theta p}B_{\theta q}G_{\theta\theta}^{-1} - B_{\theta p}G_{\theta q}G_{\theta\theta}^{-1})(\partial_\mu X^p)(\partial_\nu X^q)\\ &+ \sqrt{|h|}h^{\mu\nu}\left[G_{\theta\theta}^{-1}(\partial_\mu \tilde{X}^\theta)(\partial_\nu \tilde{X}^\theta) + 2B_{\theta q}G_{\theta\theta}^{-1}(\partial_\mu \tilde{X}^\theta)(\partial_\nu X^q)\right]\\ &+ 2\epsilon^{\mu\nu}G_{\theta q}G_{\theta\theta}^{-1}(\partial_\mu \tilde{X}^\theta)(\partial_\nu X^q) + \sqrt{|h|}R^{(2)}\Phi\Big)\end{aligned} \tag{2.39}$$

Where the term coupling the Lagrange multiplier to the gauge field has been removed by noting that the field was pure gauge. We can now recast this action to appear in a similar form to the ungauged action (2.32):

$$\begin{aligned}\tilde{S} = \frac{-1}{4\pi}\int \mathrm{d}^2\sigma\Big(&\sqrt{|h|}h^{\mu\nu}\tilde{G}_{pq}(\partial_\mu X^p)(\partial_\nu X^q) + \epsilon^{\mu\nu}\tilde{B}_{pq}(\partial_\mu X^p)(\partial_\nu X^q)\\ &+\sqrt{|h|}h^{\mu\nu}\left[\tilde{G}_{\theta\theta}(\partial_\mu \tilde{X}^\theta)(\partial_\nu \tilde{X}^\theta) + 2\tilde{G}_{\theta q}(\partial_\mu \tilde{X}^\theta)(\partial_\nu X^q)\right]\\ &+2\epsilon^{\mu\nu}\tilde{B}_{\theta q}(\partial_\mu \tilde{X}^\theta)(\partial_\nu X^q) + \sqrt{|h|}R^{(2)}\tilde{\Phi}\Big)\end{aligned} \tag{2.40}$$

These newly defined background fields, $\tilde{G}$ and $\tilde{B}$, are related to the original fields by relations known as the Buscher rules:[7]

$$\left\{\begin{array}{ccc}\tilde{G}_{\theta\theta} = \dfrac{1}{G_{\theta\theta}} & \tilde{G}_{\theta p} = \dfrac{B_{\theta p}}{G_{\theta\theta}} & \tilde{G}_{pq} = G_{pq} - \dfrac{G_{\theta p}G_{\theta q} - B_{\theta p}B_{\theta q}}{G_{\theta\theta}} \\ \\ \tilde{B}_{\theta p} = -\tilde{B}_{p\theta} = \dfrac{G_{\theta p}}{G_{\theta\theta}} & & \tilde{B}_{pq} = B_{pq} + \dfrac{G_{\theta p}B_{\theta q} - B_{\theta_p}G_{\theta q}}{G_{\theta\theta}}\end{array}\right\} \tag{2.41}$$

The dual action (2.40) is equivalent to the original action (2.32) and therefore describes the same theory, at least at a classical level. The T-duality transformation between these two theories is then:

$$\left\{X^\theta \to \tilde{X}^\theta \qquad G \to \tilde{G} \qquad B \to \tilde{B}\right\} \tag{2.42}$$

Where the dual theory is equivalent to the original through the Buscher rules. At the quantum mechanical level, the duality transformation receives corrections via the Jacobian that comes from the path-integral over the gauge field.[8] At one loop this corresponds to a shift in the dilaton field:[7]

$$\tilde{\Phi} = \Phi - \frac{1}{2}\ln G_{\theta\theta} \tag{2.43}$$

The inversion of the $G_{\theta\theta}$ component of the metric, due to the Buscher rules (2.41), is reminiscent of the radial inversion seen in 2.3. In fact, this toy model can be quickly reproduced by considering the case where $G_{\theta\theta} = R^2/\alpha'$. For simplicity, we also set $G_{\theta q} = B_{\theta q} = 0$; i.e., we demand that in the background fields there is no mixing between arbitrary directions and the gauged direction. In this case, the radial inversion $R \to \alpha'/R$ is reproduced and the duality transformation is:

$$\left\{\frac{R^2}{\alpha'} \to \frac{\alpha'}{R^2} \qquad X^\theta \to \tilde{X}^\theta \qquad G_{pq} \to G_{pq} \qquad B_{pq} \to B_{pq}\right\} \tag{2.44}$$

2.5. The Generalised Metric

Given the result of the previous section, one comes to the conclusion that the T-duality is in fact a duality between theories in different backgrounds. One may then ask what group these transformations belong to. In order to answer this question, it will be useful to first find a particular form of the theory's Hamiltonian density. We will use the standard procedure of taking the Legendre transform of the Lagrangian:

$$\mathscr{H} = \frac{\partial \mathscr{L}}{\partial \dot{X}^k}\dot{X}^k - \mathscr{L} \tag{2.45}$$

Here $\mathscr{L}$ is the Lagrangian corresponding to the action in (2.31), where for simplicity we choose the worldsheet metric to be the flat Minkowski metric. To that end, the worldsheet curvature scalar $R^{(2)}$ is zero. Such a Lagrangian takes the following form:

$$\mathscr{L} = -\frac{1}{4\pi}\Big(h^{uv}G_{ij}(\partial_\mu X^i)(\partial_\nu X^j) + \epsilon^{\mu\nu}B_{ij}(\partial_\mu X^i)(\partial_\nu X^j)\Big) \tag{2.46}$$

Where the worldsheet metric and two-dimensional Levi-Civita symbol can be represented by the following matrices:

$$h^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.47, 2.48}$$

Using these equations, the Lagrangian can be rewritten as:

$$\mathscr{L} = \frac{1}{4\pi}\Big(G_{ij}\dot{X}^i\dot{X}'^j - G_{ij}X'^iX'^j + 2B_{ij}\dot{X}^iX'^j\Big) \tag{2.49}$$

With this result, the canonical momentum is calculated to be

$$P_k \coloneqq \frac{\partial \mathscr{L}}{\partial \dot{X}^k} = \frac{1}{2\pi}\Big(G_{kj}\dot{X}^j + B_{kj}X'^j\Big) \tag{2.50}$$

For the remainder of this calculation it will be helpful to suppress the indices of these objects, as such their order becomes important. The equation for P can now be rearranged to give an expression for $\dot{X}$:

$$\dot{X} = 2\pi G^{-1}P - G^{-1}BX' \tag{2.51}$$

Using this result the Lagrangian can be rewritten in terms of P, with some simplification it becomes:

$$\mathscr{L} = \frac{1}{4\pi}\Big((2\pi)^2 P^T G^{-1} P - 2\pi P^T G^{-1} B X' \\ - 2\pi X'^T B G^{-1} P - X'^T (G - BG^{-1}B) X'\Big) \tag{2.52}$$

Combining this result with (2.50) and (2.51) allows the Hamiltonian density to be computed:

$$\mathscr{H} = \frac{1}{4\pi}\Big((2\pi)^2 P^T G^{-1} P - 2\pi P^T G^{-1} B X' \\ + 2\pi X'^T B G^{-1} P + X'^T (G - BG^{-1}B) X'\Big) \tag{2.53}$$

Finally, one may factorise this expression into the following matrix product:

$$\mathscr{H} = \frac{1}{4\pi}\begin{pmatrix} 2\pi P & X' \end{pmatrix}\begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}\begin{pmatrix} 2\pi P \\ X' \end{pmatrix} \tag{2.54}$$

The $2d \times 2d$ matrix in this expression is none other than the *generalised metric* of generalised geometry.[9] Endowed with indices, we will define this generalised metric:

$$\mathcal{H}_{MN} := \begin{pmatrix} (G^{-1})^{ij} & -(G^{-1}B)^i{}_j \\ (BG^{-1})_i{}^j & (G - BG^{-1}B)_{ij} \end{pmatrix} \tag{2.55}$$

This matrix is built from the background fields G and B, and so one may ask how it transforms under the previously seen T-duality. The transformed generalised metric is given by:

$$\tilde{\mathcal{H}} = \mathcal{O}_\theta^T \mathcal{H} \mathcal{O}_\theta \tag{2.56}$$

Where the transformation matrix, $\mathcal{O}_\theta$, is the form:[10]

$$\mathcal{O}_\theta = \begin{pmatrix} \mathbb{1} - \Theta & \Theta \\ \Theta & \mathbb{1} - \Theta \end{pmatrix} \tag{2.57}$$

Here, $\mathbb{1}$ is the d-dimensional identity matrix, and Θ is a d-dimensional matrix with zeroes everywhere except $\Theta_{\theta\theta} = 1$. This transformation yields a generalised metric built from dual fields $\tilde{G}$ and $\tilde{B}$ that are related to the original fields by the same Buscher rules as in (2.41). That is to say, the T-duality is given by this $2d$-dimensional object acting on the generalised metric. Said transformation is an $O(d,d,\mathbb{Z})$ matrix; i.e., a $2d \times 2d$ matrix, comprised of integer components, with d positive, and d negative, eigenvalues. All of the possible T-duality transformations are given by elements of $O(d,d,\mathbb{Z})$. If one were to construct a theory solely from objects invariant under $O(d,d,\mathbb{Z})$ transformations they would have a theory manifestly invariant under T-duality. $O(d,d,\mathbb{Z})$ elements obey the following condition:

$$\mathcal{O}^T \eta \mathcal{O} = \eta, \ \forall \mathcal{O} \in O(d,d,\mathbb{Z}) \tag{2.58}$$

Where η is the $O(d,d)$ metric, which can be written as:

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{2.59}$$

Aside from the T-duality, there are two more types of $O(d,d,\mathbb{Z})$ transformation that the theory is invariant under:[11]

1. The theory is invariant under $GL(d,\mathbb{Z})$ basis changes. Given such an $A \in GL(d,\mathbb{Z})$, the basis change transformation takes the form:

$$\mathcal{O}_A = \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix} \tag{2.60}$$

2. One can shift the B field by some antisymmetric matrix Ω with integer entries; i.e, $B_{ij} \to B_{ij} + \Omega_{ij}$. Such a B field transformation shifts the action by an integer multiple of 2π. Whilst shifting the action, there is no change produced in the path-integral. This B field transformation can be written as:

$$\mathcal{O}_\Omega = \begin{pmatrix} \mathbb{1} & \Omega \\ 0 & \mathbb{1} \end{pmatrix} \tag{2.61}$$

Both of these transformations obey the condition in (2.58) and so are indeed elements of $O(d,d,\mathbb{Z})$. Together with the T-duality transformations, they fully generate the $O(d,d,\mathbb{Z})$ group.[6]

2.6. Doubling the Dimensions

We now consider a space-time with multiple closed dimensions, such that for each coordinate X^i there is a corresponding winding mode ω^i and momentum mode p_i. One can then calculate the Hamiltonian:[12]

$$H = \int_0^{2\pi} \mathrm{d}\sigma \mathscr{H} \tag{2.62}$$

$$= \frac{1}{2} \begin{pmatrix} p_i & \omega^i \end{pmatrix} \begin{pmatrix} (G^{-1})^{ij} & -(G^{-1}B)^i{}_j \\ (BG^{-1})_i{}^j & (G - BG^{-1}B)_{ij} \end{pmatrix} \begin{pmatrix} p_j \\ \omega^j \end{pmatrix} + \dots \tag{2.63}$$

Here the ellipsis denotes terms not relevant to our discussion. In the above equation both the momentum and the winding appear on the same footing, with the generalised metric appearing to define some form of inner product. Just as p_i is the momentum conjugate of x^i, it is tempting to introduce a new set of dual coordinates, $\tilde{x}_i$, that the winding modes ω^i are the momentum conjugates of. In fact, this is exactly what we will do.

The presence of the generalised metric hints that there may be some generalised geometry structure to the string theory. In generalised geometry one defines a generalised tangent bundle on which vector fields and forms may be formally summed. That is, on some manifold M, with $U \in TM$ and $\chi \in T^*M$, we have $U + \chi \in TM \oplus T^*M$. We will consider a new geometry that combines a d-dimensional manifold M_d with its d-dimensional "dual manifold" $\tilde{M}_d$. This will be the space-time of the Double Field Theory (DFT), with such a $2d$-dimensional manifold defined as:

$$M_{2d} \coloneqq M_d \times \tilde{M}_d \tag{2.64}$$

In doing this we go one step further than generalised geometry. In DFT we will not only consider vector fields and forms together, we will treat them both symmetrically. The coordinates on this doubled manifold can be written as:

$$X^M \coloneqq \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \tag{2.65}$$

One could then choose a basis such that the dual coordinates $\tilde{x}_i$ are only nonzero for dimensions with isometry, however this is not necessary. Henceforth, the regular and dual coordinates will be treated on the same footing, with no mention of isometries required for the DFT to be formulated. As such, the DFT constructed will be a valid theory even when there are no closed dimensions.

The $2d$-dimensional manifold of the DFT is then endowed with an $O(d, d, \mathbb{R})$ structure. We will then construct the theory from objects with $O(d, d, \mathbb{R})$ indices such that covariant and contravariant fields are related by the $O(d, d)$ metric η. The background fields will then be described by the generalised metric, with the G and B fields that construct it now being both functions of x^i and $\tilde{x}_i$. By building the theory in this way we will construct manifestly $O(d,d,\mathbb{R})$ invariant quantities. Since the discrete T-duality group $O(d,d,\mathbb{Z})$ is a subgroup of $O(d,d,\mathbb{R})$, any manifestly $O(d,d,\mathbb{R})$ invariant quantities will also have T-duality as a manifest symmetry.

Doubling the dimensions in this way follows a similar approach to that of Kaluza-Klein theory. We will see that the diffeomorphisms on the original d-dimensional space-time, and the two-form gauge transformations of the B field, will be combined into generalised diffeomorphisms on the doubled manifold. In this way the DFT will not only make T-duality a manifest symmetry, but it will also combine the local transformations of the closed string theory into one type of diffeomorphism.

However, at present it has not been possible to construct such a DFT without introducing an additional constraint, known as the *strong constraint*. This will be further discussed later in section 3.4.

3. The $O(d,d,\mathbb{R})$ Duality Group

In this chapter the $O(d,d,\mathbb{R})$ group, hereby referred to as just the $O(d,d)$ group, is examined. This group describes the structure of the DFT, and so we will begin by discussing some of its properties. After this the tools with which the DFT is written will be presented. This then leads to a brief analysis of how one constructs a theory that is manifestly invariant under the $O(d,d)$ group, and hence the T-duality. Finally, we discuss the constraints imposed in DFT, and how they relate the doubled theory back to the original d-dimensional one.

3.1. Properties of $O(d,d)$

We will now explore some of the properties exhibited by elements of the $O(d,d)$ group. These elements obey the same condition, (2.58), as elements of the discrete $O(d,d,\mathbb{Z})$ group, that is:

$$h^T \eta h = \eta, \ \forall h \in O(d,d) \tag{3.1}$$

This expression can be rearranged to give an equation for the inverse of any arbitrary $O(d,d)$ element:

$$\eta h^T \eta = h^{-1} \tag{3.2}$$

In general, any element of $O(d,d)$ can be written in terms of four d-dimensional matrices, here denoted by a, b, c, and d, such that:

$$h := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.3}$$

Given this expression, together with (3.2), we will be able to calculate the inverse of any $O(d,d)$ matrix:

$$h^{-1} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.4}$$

$$= \begin{pmatrix} d^T & b^T \\ c^T & a^T \end{pmatrix} \tag{3.5}$$

Using this result one can then consider what conditions are placed on the d-dimensional matrices from which an $O(d,d)$ element is built. By requiring the definition of the matrix inverse, $hh^-1 = \mathbb{1}$, the following conditions are found:[13]

$$a^T c + c^T a = 0 \tag{3.6}$$

$$b^T d + d^T b = 0 \tag{3.7}$$

$$a^T d + c^T b = \mathbb{1} \tag{3.8}$$

It can be seen that the generalised metric itself too satisfies equation (3.1). It follows then that $\mathcal{H} \in O(d,d)$, and so the inverse of the generalised metric can be easily calculated:

$$(\mathcal{H}^{-1})^{MN} = \eta^{MP}\mathcal{H}_{PQ}\eta^{QN} \tag{3.9}$$

$$= \begin{pmatrix} (G - BG^{-1}B)_{ij} & (BG^{-1})_i{}^j \\ -(G^{-1}B)^i{}_j & (G^{-1})^{ij} \end{pmatrix} \tag{3.10}$$

Here the generalised metric's inverse, $\mathcal{H}^{MN}$, was found by using the $O(d,d)$ metric η to raise it's indices. It then seems fitting to call these indices "$O(d,d)$ indices". It's important to note that the generalised metric is symmetric, and so in the above calculation we did not need to take its transpose. In general, $O(d,d)$ matrices need not be symmetric, and so the transpose present in (3.2) is indeed required.

3.2. $O(d,d)$ Fundamental Objects for DFT

In this section we will construct some useful DFT objects. These objects will be referred to as "$O(d,d)$ fundamental" objects, since they transform in the fundamental representation of $O(d,d)$.[12] That is, an arbitrary $O(d,d)$ fundamental object, U_M, transforms as:

$$U_M \to h_M{}^N U_N \tag{3.11}$$

Where the transformation is parametrised by some $O(d, d)$ matrix, h. One can also raise and lower the indices of these fundamental objects using the $O(d, d)$ metric; i.e., $U^M = \eta^{MN} U_N$. Such objects are built from $2d$ components, half of which correspond to differential one-form components and half to vector elements. In other words, these objects have both covariant and contravariant components: $\bullet^M = (\bullet_i, \bullet^i)$ and $\bullet_M = (\bullet^i, \bullet_i)$.[14]

We have already encountered one of these $O(d, d)$ fundamental objects: the coordinate X^M defined in (2.65). One may then ask if it is possible to construct a fundamental object from partial derivatives; this is indeed possible and can be written as:

$$\partial^M := \begin{pmatrix} \partial_i \\ \tilde{\partial}^i \end{pmatrix} \tag{3.12}$$

Here $\tilde{\partial}^i$ corresponds to a partial derivative with respect to the dual coordinate $\tilde{x}_i$.

Another example of a fundamental object is the vector made from momentum and winding modes seen in (2.63). This object is given by:[12]

$$Z^M := \begin{pmatrix} p_i \\ \omega^i \end{pmatrix} \tag{3.13}$$

In a sense, this is the object built from the momentum conjugates of the coordinates X^M.

Thus, we are now able to write coordinates, partial derivatives, and momenta in DFT. We also have the generalised metric to describe the background fields of the theory. In addition to these objects, we will need to provide a scalar dilaton for the DFT. Just as the dilaton is shifted in T-duality, the doubled dilaton will also be shifted. It can be written as:[11]

$$d = \Phi - \frac{1}{2} \ln \sqrt{|G|} \tag{3.14}$$

This dilaton is invariant under the T-duality shift of Φ seen in (2.43). The DFT action considered later, in chapter 5, will be written in terms of these objects given here.

3.3. $O(d, d)$ Transformations

In order to construct a manifestly $O(d, d)$ invariant theory we will need to be able to write quantities in terms of $O(d, d)$ scalars. An $O(d, d)$ scalar is any object, constructed from $O(d, d)$ elements and fundamental objects, with all of its $O(d, d)$ indices fully contracted. An example of such a scalar is $U^M V_M$, where U and V are both fundamental objects. It will now be shown that this object, whose indices are fully contracted, is indeed invariant under some $O(d,d)$ transformation h:

$$U^M V_M \to U^P h_P{}^M h^Q{}_M V_Q \tag{3.15}$$
$$= U^P V_Q h_{PR} \eta^{RM} \eta_{MS} h^{SQ} \tag{3.16}$$
$$= U^P V_Q h_{PR} \delta^R{}_S h^{SQ} \tag{3.17}$$
$$= U^P V_Q h_{PS} h^{SQ} \tag{3.18}$$
$$= U^P V_Q \delta_P{}^Q \tag{3.19}$$
$$= U^P V_P \tag{3.20}$$

This is a simple calculation, however it has a very important implication. It shows us that pairs of contracted $O(d,d)$ indices transform in such a way that the transformations cancel. It follows that any object with fully contracted $O(d,d)$ indices is invariant under $O(d,d)$ transformations.

One such invariant quantity is the Hamiltonian calculated in (2.63). It is useful at this point to define a single d-dimensional quantity combining the background fields: $E := G + B$. Hence one could write the generalised metric as a function of this quantity, $\mathcal{H}(E)$. How do transformations of E appear in the DFT? We now consider a change in this background field $E \to E'$ such that the values of the momentum and winding numbers in Z are unchanged but reshuffled in some way. We then require that the Hamiltonian remain invariant under this reshuffling:[12]

$$Z^M \mathcal{H}(E)_{MN} Z^N = Z'^P \mathcal{H}(E')_{PQ} Z'^Q \tag{3.21}$$

The reshuffling of Z can in fact be described by an $O(d,d)$ matrix, so we have:

$$Z'^P h_P{}^M \mathcal{H}(E)_{MN} h^N{}_Q Z'^Q = Z'^P \mathcal{H}(E')_{PQ} Z'^Q \tag{3.22}$$

Thus, we identify that the generalised metric of the transformed background is:

$$\mathcal{H}(E')_{PQ} = h_P{}^M \mathcal{H}(E)_{MN} h^N{}_Q \tag{3.23}$$

We then see that this change in the background E is generated by the action of some $O(d,d)$ transformation on the generalised metric. Hence, one can use the $O(d,d)$ elements to transform E, and one may then ask exactly how this field transforms. It can be shown that the change in E due to h is given by the fractional linear transformation:[11]

$$E' = h(E) = (aE + b)(cE + d)^{-1} \tag{3.24}$$

Where a, b, c, and d, are the d-dimensional matrices from which h is built, as seen in equation (3.3).

3.4. Constraints in DFT

3.4.1. The Weak Constraint

There are two constraints that we need to consider in DFT. The first of which is the *weak constraint* which follows from the standard level matching constraint of string theory. This constrain expressed in terms of the Virasoro operators is:

$$L_0^\perp - \bar{L}_0^\perp = 0 \tag{3.25}$$

These Virasoro operators can then be expressed in terms of the zero modes in the standard way:

$$L_0^\perp = \frac{1}{2}\alpha_0^i G_{ij} \alpha_0^j + N^\perp \tag{3.26}$$
$$\bar{L}_0^\perp = \frac{1}{2}\bar{\alpha}_0^i G_{ij} \bar{\alpha}_0^j + \bar{N}^\perp \tag{3.27}$$

In order to calculate these operators we need first an expression for the zero modes of the theory. In terms of the momentum and the winding these are:[15]

$$\alpha_0^i = \frac{1}{\sqrt{2}} G^{ij}(p_j - E_{jk}\omega^k) \tag{3.30}$$
$$\bar{\alpha}_0^i = \frac{1}{\sqrt{2}} G^{ij}(p_j + E^T_{jk}\omega^k) \tag{3.31}$$

These expressions can then be rewritten using the representations of the momentum and winding as partial derivatives; i.e., $p_j = -i\partial)j$ and $\omega^k = -i\tilde{\partial}^k$. This yields:

$$\alpha_0^i = -\frac{i}{\sqrt{2}} G^{ij}(\partial_j - E_{jk}\tilde{\partial}^k) \tag{3.30}$$
$$\bar{\alpha}_0^i = -\frac{i}{\sqrt{2}} G^{ij}(\partial_j + E^T_{jk}\tilde{\partial}^k) \tag{3.31}$$

With these in place the difference of the two Virasoro operators can now be evaluated:

$$\begin{aligned} L_0^\perp - \bar{L}_0^\perp = N^\perp - \bar{N}^\perp - \frac{1}{4}\Big(& G^{ik}(\partial_k - E_{kl}\tilde{\partial}^l) G_{ij} G^{jm}(\partial_m - E_{mn}\tilde{\partial}^n) \\ & - G^{ik}(\partial_k + E^T_{kl}\tilde{\partial}^l) G_{ij} G^{jm}(\partial_m + E^T_{mn}\tilde{\partial}^n)\Big) \end{aligned} \tag{3.32}$$

After much simplification, and expanding out E in terms of G and B, this equation then becomes:

$$L_0^\perp - \bar{L}_0^\perp = N^\perp - \bar{N}^\perp + \partial_i \tilde{\partial}^i \tag{3.33}$$

By applying the level matching condition, (3.25), this becomes:

$$N^\perp - \bar{N}^\perp = -\partial_i \tilde{\partial}^i \tag{3.34}$$

Hence, for fields with $N^\perp = \bar{N}^\perp$, such as the massless fields G, B and Φ, the condition becomes:

$$\partial^M \partial_M \psi = 0 \tag{3.35}$$

Where ψ is such a field with $(N^\perp - \bar{N}^\perp)\psi = 0$ and we have identified that $2\partial_i \tilde{\partial}^i = \partial^M \partial_M$. This constraint is known as the weak constraint. It follows from the standard level-matching constraint and so it is not surprising to see it in DFT.

3.4.2. The Strong Constraint

The second constraint, known as the *strong constraint*, is somewhat more curious. The strong constraint demands that (3.35) be true when $\psi = \psi_1 \psi_2$; i.e., it demands that $\partial^M \partial_M$ acting on products of fields is zero. Written, this constraint is:

$$\partial^M \partial_M \psi_1 \psi_2 = 0 \tag{3.36}$$

Which, given the weak constraint, is equivalent to saying:

$$(\partial^M \psi_1)(\partial_M \psi_2) = 0 \tag{3.37}$$

This strong constraint does not follow from a previous result, instead we impose it. In order to solve this constraint one chooses that the fields are functions of just d of the possible $2d$ coordinates. Typically this is done one of two ways: either setting $\partial_i = 0$ or setting $\tilde{\partial}_i = 0$.

This constraint is incredibly restricting and ideally one would construct a theory where it is relaxed. However, in chapter 4 we will see that the strong constraint is necessary for DFT, without it the local symmetries of the theory cannot be properly described.

3.4.3. The Section Condition

One may ask how the $2d$-dimensional DFT can be related back to the original d-dimensional theory. This is realised through imposing the *section condition*. The section condition restricts the theory to a maximally isotropic d-dimensional subspace of the $2d$-dimensional DFT space-time.[16] In essence, it reduces the theory to one in which there is only dependence on half of the coordinates. This is reminiscent of the strong constraint, which provides a canonical choice for the section condition. Solving the strong constraint, by demanding that partial derivatives along d of the $2d$ dimensions are zero, also imposes the section condition.

If one can reduce the DFT to a d-dimensional theory, then how are the T-dual theories found? The T-duality is a manifest symmetry of the DFT and so one expects to be able to attain all of the dual theories from it. This is indeed the case. In backgrounds with isometry, the duality related theories are given by the different ways of imposing the section condition.[16] For example, consider a $2d$-dimensional space-time with coordinates $(y, \tilde{y}, x^1, \ldots, x^{d-1}, \tilde{x}^1, \ldots, \tilde{x}^{d-1})$, where y denotes a direction with isometry and $\tilde{y}$ its dual. Including y and $\tilde{y}$ in a d-dimensional theory results in a redundancy of coordinates, since both describe the same isometry. In this case, the T-dual theories are related by section conditions that choose either y or $\tilde{y}$ as null directions. Upon realisation of the section condition the local $O(d, d)$ symmetries break, leaving just the global $O(d, d, \mathbb{Z})$ transformations describing the T-duality.

4. Local Transformations in DFT

Aside from the now manifest $O(d, d)$ duality, this bosonic formulation of DFT has two more symmetries:[17]

- Diffeomorphisms of the fields G and B (and d).
- Gauge transforms of the two-form B.

Let U be a vector field generating such a diffeomorphism, and χ be a one-form generating a gauge transformation, such that

$$G_{ij} \to G_{ij} + \mathcal{L}_U G_{ij} \tag{4.1}$$

$$B_{ij} \to B_{ij} + \mathcal{L}_U B_{ij} + \partial_i \chi_j - \partial_j \chi_i \tag{4.2}$$

are the background fields after such infinitesimal transformations. Here $\mathcal{L}_U$ is the Lie derivative with respect to U. In this section these local transformations will be explored. An understanding of these local symmetries is crucial for calculating many quantities, such as the conserved currents of the theory.

4.1. The Lie Deriviative

The Lie derivative $\mathcal{L}$ measures how a tensor field changes along the *flow* of some vector field. One can define a continuous one-parameter group of diffeomorphisms along some vector field U:

$$\begin{aligned} \varphi_\xi : \; M &\to M \\ \hat{x} &\mapsto \varphi_\xi(\hat{x}) \end{aligned} \tag{4.3}$$

Where the diffeomorphism φ is parametrised by the continuous parameter ξ, and M is a differentiable manifold endowed with coordinates x^i at each point $\hat{x}$, such that:

$$\varphi_\xi(x^i) = x^i + \xi U^i \tag{4.4}$$

We say that the vector field U *generates* these diffeomorphisms. The Lie derivative of some vector field V along U, where U and V are both elements of the tangent bundle TM, can now be defined as:[18]

$$\mathcal{L}_U V \coloneqq \lim_{\xi \to 0} \frac{1}{\xi} [(\varphi_{-\xi})_* \, V|_{\varphi_\xi(\hat{x})} - V|_{\hat{x}}] \tag{4.5}$$

With $(\varphi_{-\xi})_*$ denoting the pushforward from the tangent space $T_{\varphi_\xi(\hat{x})}M$ to $T_{\hat{x}}M$. Written in this way, the Lie derivative is the difference between V evaluated at $\varphi_\xi(\hat{x})$ and at $\hat{x}$. It is the change in V due to an infinitesimal diffeomorphism along U. Utilising the pushforward means that the Lie derivative is well-defined on any differentiable manifold; additional structures such as a metric connection are not necessary. The Lie derivative of an arbitrary rank (n, m) tensor field can be written:

$$\mathcal{L}_U T^{\mu_1 \ldots \mu_n}_{\sigma_1 \ldots \sigma_m} = U^\rho (\partial_\rho T^{\mu_1 \ldots \mu_n}_{\sigma_1 \ldots \sigma_m}) + \sum_{i=1}^{m} (\partial_{\sigma_i} U^\rho) T^{\mu_1 \ldots \mu_n}_{\sigma_1 \ldots \rho \ldots \sigma_m} - \sum_{j=1}^{n} (\partial_\rho U^{\mu_j}) T^{\mu_1 \ldots \rho \ldots \mu_n}_{\sigma_1 \ldots \sigma_m} \tag{4.6}$$

A full derivation of this result can be found in Appendix A.

The Lie derivative is a linear operator, and as a derivation it obeys a Leibniz rule:

$$\mathcal{L}_U(AB) = A(\mathcal{L}_U B) + (\mathcal{L}_U A)B \tag{4.7}$$

Where A and B are arbitrary tensor fields. When equipped with a Lie bracket

$$\begin{aligned} [\,\cdot\,, \,\cdot\,] : \; & L \times L \to L \\ & \mathcal{L}_U \times \mathcal{L}_V \mapsto [\mathcal{L}_U, \mathcal{L}_V] \end{aligned} \tag{4.8}$$

the Lie derivatives along vector fields form a Lie algebra. Here, $\mathcal{L}_U$ and $\mathcal{L}_V$ are Lie derivatives along arbitrary vector fields U and V respectively. Such an algebra then obeys the following relations, $\forall \mathcal{L}_U, \mathcal{L}_V, \mathcal{L}_W \in L$:

$$[\mathcal{L}_U, \mathcal{L}_V] = -[\mathcal{L}_V, \mathcal{L}_U] \tag{4.9}$$

$$[\mathcal{L}_U, \mathcal{L}_V] = \mathcal{L}_{[U,V]} \tag{4.10}$$

$$[\mathcal{L}_U, [\mathcal{L}_V, \mathcal{L}_W]] + [\mathcal{L}_V, [\mathcal{L}_W, \mathcal{L}_U]] + [\mathcal{L}_W, [\mathcal{L}_U, \mathcal{L}_V]] = 0 \tag{4.11}$$

Where (4.9) is the bracket's antisymmetry property, (4.10) its closure written as a Lie algebra homomorphism, and (4.11) a Jacobi identity.

4.2. Kaluza-Klein Theory as an Analogy

Consider now a Kaluza-Klein theory (KK theory) of a five-dimensional space-time, one dimension of which is compact. Ignoring the dilaton field, such a space-time is described by the metric:

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ \hline A_\nu & 1 \end{array}\right) \tag{4.12}$$

Where the "hatted" indices run over all space-time coordinates, and the unhatted indices label just the noncompactified dimensions. A is typically identified with the electromagnetic four-potential (when the compact dimension has U(1) isometry). This is strikingly similar to DFT's generalised metric.[19] This is not so surprising when one recalls that we began by considering space-times with toroidal geometry. In this analogy, DFT looks like a $2d$-dimensional KK theory with:

- d compact dimensions, their geometry described by the inverse metric G^{-1}.
- The antisymmetric two-form B taking the place of the one-form A.

Equipped with this analogy, it becomes a point of interest to consider how transformations behave in KK theory.

Let's look at the Lie derivative of the KK metric taken along some vector field $\hat{U}$ with:

$$\hat{U}^{\hat{\mu}} = \begin{pmatrix} U^\mu \\ \chi \end{pmatrix} \tag{4.13}$$

Using equation (4.6) we have:

$$\mathcal{L}_{\hat{U}}\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{U}^{\hat{\rho}}(\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}}) + (\partial_{\hat{\mu}}\hat{U}^{\hat{\rho}})\hat{g}_{\hat{\rho}\hat{\nu}} + (\partial_{\hat{\nu}}\hat{U}^{\hat{\rho}})\hat{g}_{\hat{\mu}\hat{\rho}} \tag{4.14}$$

$$= U^\rho(\partial_\rho\hat{g}_{\hat{\mu}\hat{\nu}}) + (\partial_{\hat{\mu}}U^\rho)\hat{g}_{\rho\hat{\nu}} + (\partial_{\hat{\mu}}\chi)\hat{g}_{5\hat{\nu}} + (\partial_{\hat{\nu}}U^\rho)\hat{g}_{\hat{\mu}\rho} + (\partial_{\hat{\nu}}\chi)\hat{g}_{\hat{\mu}5} \tag{4.15}$$

Where we have demanded that derivatives with respect to the compact dimension are zero; i.e., $\partial_5 = 0$. This is known as *Kaluza-Klein reduction*; it is the requirement that there is no dependence on the compact dimension. By choosing $\hat{\mu} = 5$ and $\hat{\nu} = \nu$ we can calculate $\mathcal{L}_{\hat{U}}A_\nu$:

$$\mathcal{L}_{\hat{U}}\hat{g}_{5\nu} = \mathcal{L}_{\hat{U}}A_\nu = U^\rho(\partial_\rho A_\nu) + (\partial_\nu U^\rho)A_\rho + (\partial_\nu\chi)\hat{g}_{55} \tag{4.16}$$

$$= \mathcal{L}_U A_\nu + \partial_\nu\chi \tag{4.17}$$

Hence, we have found that the Lie derivative of A along $\hat{U}$ is a combination of the Lie derivative along U and the infinitesimal gauge transformation generated by χ. Returning now to (4.15) we can set $\hat{\mu} = \mu$ and $\hat{\nu} = \nu$:

$$\begin{aligned}\mathcal{L}_{\hat{U}}\hat{g}_{\mu\nu} = U^\rho(\partial_\rho g_{\mu\nu}) + (\partial_\mu U^\rho)g_{\rho\nu} + (\partial_\nu U^\rho)g_{\mu\rho} + U^\rho(\partial_\rho A_\mu A_\nu) \\ + (\partial_\mu U^\rho)A_\rho A_\nu + (\partial_\mu\chi)A_\nu + (\partial_\nu U^\rho)A_\mu A_\rho + (\partial_\nu\chi)A_\mu\end{aligned} \tag{4.18}$$

This can then be rewritten in terms of the Lie derivatives:

$$\mathcal{L}_{\hat{U}}\hat{g}_{\mu\nu} = \mathcal{L}_U g_{\mu\nu} + (\mathcal{L}_U A_\mu + \partial_\mu\chi)A_\nu + (\mathcal{L}_U A_\nu + \partial_\nu\chi)A_\mu \tag{4.19}$$

Thus we see that, under KK reduction, diffeomorphisms in the five-dimensional geometry can be decomposed into diffeomorphisms in the standard four-dimensional geometry and gauge transformations of the field A. More astutely, we find that the group of diffeomorphisms on the open space-time and the group of gauge transformations of A are both subgroups of the group of diffeomorphisms on the entire five-dimensional space-time. One would then be keen to see an equivalent result for diffeomorphisms in DFT.

4.3. The Generalised Lie Derivative

In (4.2) we see that gauge transformations of the Kalb-Ramond B field are generated by one-forms. Clearly then, if diffeomorphisms and gauge transformations in DFT are to be described by a single Lie derivative, we need to be able to take Lie derivatives along both vector fields and one-forms. Consider then an object, of the type of those discussed in section 3.2, containing both one-form and vector elements:*

$$\hat{U}^M = \begin{pmatrix}\chi_i \\ U^i\end{pmatrix} \tag{4.20}$$

In DFT we are equipped to take partial derivatives of these arbitrary objects, so naïvely we attempt to take the Lie derivative of some vector field V^M along $\hat{U}^M$:

$$\mathcal{L}_{\hat{U}}V^M = \hat{U}^P(\partial_P V^M) - (\partial_P\hat{U}^M)V^P \tag{4.21}$$

The issue with this arises when we choose the case where $\hat{U}^M = \partial^M\chi$; i.e., the case where we take a Lie derivative along some trivial parameter.[12] This should be zero, yet it is

*The hat on $\hat{U}^M$ is superficially chosen to allow it to be distinguished from the d-dimensional vector field U^i. This will prove useful in 4.4 and 4.5 where both objects appear with indices suppressed.

not:

$$\mathcal{L}_{\partial\chi} V^M = \partial^P \chi (\partial_P V^M) - (\partial_P \partial^M \chi) V^P \neq 0 \tag{4.22}$$

The first term vanishes due to the strong constraint, but the second term does not. One can propose an ansatz for a *generalised Lie derivative* to remedy this problem:[11]

$$\hat{\mathcal{L}}_{\hat{U}} V^M := \mathcal{L}_{\hat{U}} V^M + Y^{MN}{}_{PQ} (\partial_N \hat{U}^P) V^Q \tag{4.23}$$

One sees that setting $Y^{MN}{}_{PQ} = \eta^{MN} \eta_{PQ}$ immediately deals with the issue met in (4.22):

$$\hat{\mathcal{L}}_{\partial\chi} V^M = \partial^P \chi (\partial_P V^M) - (\partial_P \partial^M \chi) V^P + (\partial^M \partial_P \chi) V^P \tag{4.24}$$

Which is clearly zero since partial derivatives commute.

At first may seem somewhat convoluted to have utilised such an ansatz when this derivative could likely be defined by inspection. However, in theories describing additional dualities, such as S-dualities, the same ansatz appears but with different values of $Y^{MN}{}_{PQ}$. The value of $Y^{MN}{}_{PQ}$ required here is specific to the $O(d,d)$ symmetry group.

As with (4.6), one can write the generalised Lie derivative of an arbitrary rank (n,m) tensor field:

$$\hat{\mathcal{L}}_{\hat{U}} T^{N_1 \dots N_n}{}_{M_1 \dots M_m} = \mathcal{L}_{\hat{U}} T^{N_1 \dots N_n}{}_{M_1 \dots M_m} - \sum_{i=1}^{m} (\partial^P \hat{U}_{M_i}) T^{N_1 \dots N_n}{}_{M_1 \dots P \dots M_m} + \sum_{j=1}^{n} (\partial^{N_j} \hat{U}_P) T^{N_1 \dots P \dots N_n}{}_{M_1 \dots M_m} \tag{4.25}$$

4.4. Generalised Diffeomorphisms in DFT

Armed with the generalised Lie derivative, we can now attempt the DFT equivalent of the calculation seen in 4.2. By using (4.25) we find that the generalised Lie derivative of the generalised metric, taken along an object of the same type as defined in (4.20), is:

$$\hat{\mathcal{L}}_{\hat{U}} \mathcal{H}_{MN} = \mathcal{L}_{\hat{U}} \mathcal{H}_{MN} - (\partial^P \hat{U}_M) \mathcal{H}_{PN} - (\partial^P \hat{U}_N) \mathcal{H}_{MP} \tag{4.26}$$

Choosing $M = i$ and $N = j$, where i and j label the first d components, allows us to calculate $\hat{\mathcal{L}}_{\hat{U}} G^{ij}$:

$$\begin{aligned} \hat{\mathcal{L}}_{\hat{U}} G^{ij} = (U^k \partial_k + \chi_k \tilde{\partial}^k) G^{ij} + (\tilde{\partial}^i \chi_k) G^{kj} + (\tilde{\partial}^i U^k) B_{kl} G^{lj} \\ + (\tilde{\partial}^j \chi_k) G^{ik} - (\tilde{\partial}^j U^k) G^{il} B_{lk} - (\partial_k U^i) G^{kj} \\ - (\tilde{\partial}^k U^i) B_{kl} G^{lj} - (\partial_k U^j) G^{ik} + (\tilde{\partial}^k U^j) G^{il} B_{lk} \end{aligned} \tag{4.27}$$

The DFT analogue of performing a KK reduction is to solve the section condition. Here we will solve this constraint by requiring there be no dependence on the dual coordinates; i.e, $\tilde{\partial} = 0$. Applying this condition yields:

$$\begin{aligned} \hat{\mathcal{L}}_{\hat{U}} G^{ij} &= U^k(\partial_k G^{ij}) - (\partial_k U^i) G^{kj} - (\partial_k U^j) G^{ik} && (4.28) \\ &= \mathcal{L}_U G^{ij} && (4.29) \end{aligned}$$

Thus we see that, upon realising the section condition, generalised Lie derivative of G^{ij} along $\hat{U}$ is just the usual Lie derivative of G^{ij} taken along U.

Choosing the case where M runs only over the $d+1$ to $2d$ components and N runs over the first d components allows the calculation of $\hat{\mathcal{L}}_{\hat{U}}(B_{ik}G^{kj})$:

$$\begin{aligned} \hat{\mathcal{L}}_{\hat{U}}(B_{ik}G^{kj}) = (U^k\partial_k + \chi_k\tilde{\partial}^k)B_{il}G^{lj} + (\partial_i\chi_k)G^{kj} + (\partial_i U^k)B_{kl}G^{lj} \\ + (\tilde{\partial}^j\chi_k)B_{il}G^{lk} + (\tilde{\partial}^j U^k)(G_{ik} - B_{il}G^{lm}B_{mk}) - (\partial_k\chi_i)G^{kj} \\ - (\tilde{\partial}^k\chi_i)B_{kl}G^{lj} - (\partial_k U^j)B_{il}G^{lk} + (\tilde{\partial}^k U^j)(G_{ik} - B_{il}G^{lm}B_{mk}) \end{aligned} \tag{4.30}$$

By applying the section condition we find:

$$\begin{aligned} \hat{\mathcal{L}}_{\hat{U}}(B_{ik}G^{kj}) = U^k(\partial_k B_{il}G^{lj}) + (\partial_i\chi_k)G^{kj} + (\partial_i U^k)B_{kl}G^{lj} \\ - (\partial_k\chi_i)G^{kj} - (\partial_k U^j)B_{il}G^{lk} \end{aligned} \tag{4.31}$$

After some relabelling, and expanding the first term using the product rule, this can be rewritten as:

$$\begin{aligned} \hat{\mathcal{L}}_{\hat{U}}(B_{ik}G^{kj}) = [U^l(\partial_l B_{ik}) + (\partial_i U^l)B_{lk} + \partial_i\chi_k - \partial_k\chi_i]G^{kj} \\ + B_{ik}[U^l(\partial_l G^{kj} - (\partial_l U^j)G^{kl}] \end{aligned} \tag{4.32}$$

Ideally we would like to identify terms here with Lie derivatives, however the following two terms are missing:

$$(\partial_k U^l)B_{il}G^{kj} - (\partial_l U^k)B_{ik}G^{lj} \tag{4.33}$$

Fortunately, both of these terms are identical up to a relabelling of indices and so sum to zero. As such, we can add them to (4.32) for "free":

$$\begin{aligned} \hat{\mathcal{L}}_{\hat{U}}(B_{ik}G^{kj}) = [U^l(\partial_l B_{ik}) + (\partial_i U^l)B_{lk} + (\partial_k U^l)B_{il} + \partial_i\chi_k - \partial_k\chi_i]G^{kj} \\ + B_{ik}[U^l(\partial_l G^{kj} - (\partial_l U^j)G^{kl} - (\partial_l U^k)G^{lj}] \end{aligned} \tag{4.34}$$

We can now begin identifying terms with Lie derivatives:

$$\hat{\mathcal{L}}_{\hat{U}}(B_{ik}G^{kj}) = (\mathcal{L}_U B_{ik} + \partial_i \chi_k - \partial_k \chi_i)G^{kj} + B_{ik}\mathcal{L}_U G^{kj} \tag{4.35}$$

The left-hand side of this equation can then be expanded using a Leibniz rule to identify specifically that:

$$\hat{\mathcal{L}}_{\hat{U}}(B_{ik}) = \mathcal{L}_U B_{ik} + \partial_i \chi_k - \partial_k \chi_i \tag{4.36}$$

Remarkably this result mirrors that found in KK theory: The generalised Lie derivative on the doubled space-time describes both diffeomorphisms of the fields G and B as well as the gauge transformations of the B field.† We say that the generalised Lie derivative combines them both into one type of transformation, *generalised diffeomorphisms*. These generalised diffeomorphisms form the local symmetries of DFT, and the generalised Lie derivative provides a single operator to describe them.

4.5. Properties of the Generalised Lie Derivative

4.5.1. A Bracket for Generalised Lie

Equipped now with a generalised Lie derivative, one may ask if there an appropriate choice of Lie bracket to pair with such a derivative. In other words, what bracket suitably describes the algebra of these generalised diffeomorphisms? It is sensible then to begin by investigating the infinitesimal changes seen in (4.1) and (4.2):

$$\delta_{U+\chi} G = \mathcal{L}_U G \tag{4.37}$$

$$\delta_{U+\chi} B = \mathcal{L}_U B + \mathrm{d}\chi \tag{4.38}$$

Where the gauge transformation generated by the one-form χ has been written as an exterior derivative. Here the formal sum of the vector field U and one-form χ is defined on some manifold M for $U + \chi \in TM \oplus T^*M$, with $U \in TM$ and $\chi \in T^*M$. The gauge algebra is then described by the commutator of these transformations:[12]

$$[\delta_{U_1+\chi_1}, \delta_{U_2+\chi_2}]G = \mathcal{L}_{[U_1,U_2]}G \tag{4.39}$$

$$[\delta_{U_1+\chi_1}, \delta_{U_2+\chi_2}]B = \mathcal{L}_{[U_1,U_2]}B + \mathrm{d}(\mathcal{L}_{U_1}\chi_2 - \mathcal{L}_{U_2}\chi_1) \tag{4.40}$$

†The calculation done here is actually for G^{-1} not G. A straightforward calculation of $\hat{\mathcal{L}}_{\hat{U}}\mathcal{H}_{MN}$ for the case where M and N both run over the $d+1$ to $2d$ components yields an equivalent result for G.

Leading us to conclude that in general:

$$[\delta_{U_1+\chi_1}, \delta_{U_2+\chi_2}] = \delta_{[U_1,U_2]+\mathcal{L}_{U_1}\chi_2-\mathcal{L}_{U_2}\chi_1} \tag{4.41}$$

This provides a natural choice of bracket for elements on $TM \oplus T^*M$:

$$[U_1 + \chi_1, U_2 + \chi_2] = [U_1, U_2] + \mathcal{L}_{U_1}\chi_2 - \mathcal{L}_{U_2}\chi_1 \tag{4.42}$$

We see in (4.40) that the transformation generating forms always appear as an exterior derivative. Thus, the gauge algebra is unchanged by the introduction of an arbitrary exact form, leading us to conclude that in general:[20]

$$[U_1 + \chi_1, U_2 + \chi_2]_\kappa = [U_1, U_2] + \mathcal{L}_{U_1}\chi_2 - \mathcal{L}_{U_2}\chi_1 - \frac{\kappa}{2}\,\mathrm{d}(\iota_{U_1}\chi_2 - \iota_{U_2}\chi_1) \tag{4.43}$$

Where κ is some arbitrary constant and ι_U denotes an inner derivative defined by the vector U. One sees that the introduction of this term only generates trivial transformations (proportional to $\mathrm{d}^2 = 0$).

We will choose the case where $\kappa = 1$ for which (4.43) becomes the *Courant bracket* of generalised geometry, first presented by Theodore Courant in 1990.[21]

$$[U_1 + \chi_1, U_2 + \chi_2]_{\mathrm{Cour}} \coloneqq [U_1, U_2] + \mathcal{L}_{U_1}\chi_2 - \mathcal{L}_{U_2}\chi_1 - \frac{1}{2}\,\mathrm{d}(\iota_{U_1}\chi_2 - \iota_{U_2}\chi_1) \tag{4.44}$$

A straightforward calculation shows that the $\kappa = 0$ bracket obeys a Jacobi identity, since it is a linear combination of terms already known to satisfy a Jacobi identity. The Courant bracket, however, does not satisfy a Jacobi identity (this will be seen explicitly in 4.5.3). Thus, it doesn't form a Lie algebra, instead it is said to form a Courant algebroid.[22] One may then ask why $\kappa = 1$ is a good choice, the answer to which is that the Courant bracket is a special case with an additional symmetry: It can be seen that the so called *B-field transformations* are automorphisms of the Courant bracket.[23] That is, given the definition of the B-field transformation[12]

$$U + \chi \to U + \chi + \iota_U B, \tag{4.45}$$

the Courant bracket of two transformed elements is the B-field transformation of the Courant bracket. That is to say:

$$\begin{aligned} [U_1 + \chi_1, U_2 + \chi_2]_{\mathrm{Cour}} &\to [U_1 + \chi_1 + \iota_{U_1}B,\, U_2 + \chi_2 + \iota_{U_2}B]_{\mathrm{Cour}} \\ &= [U_1 + \chi_1, U_2 + \chi_2]_{\mathrm{Cour}} + \iota_{[U_1,U_2]}B \end{aligned} \tag{4.46}$$

Where B is any closed two-form, and is not to be confused with the Kalb-Ramond field.[‡] The result stated in equation (4.46) is shown to be true in Appendix B.

In DFT the background geometry is doubled, setting both vectors and forms on the same footing. Whilst the generalised geometry setup does allow vectors and forms to be formally summed its Courant bracket does not treat them symmetrically. This can be quickly seen by realising that $[U_1, U_2]_{\text{Cour}} \neq 0$ whilst $[\chi_1, \chi_2]_{\text{Cour}} = 0$. A bracket of two DFT parameters, $\hat{U}_1$ and $\hat{U}_2$ defined in the same way as (4.20), would need to treat both forms and vectors identically. To describe the algebra of generalised diffeomorphisms an $O(d,d)$ covariant *C-bracket* $[\,\cdot\,,\,\cdot\,]_C$ is defined:[14]

$$\begin{aligned}([\hat{U}_1, \hat{U}_2]_C)^M \coloneqq \hat{U}_1^P \partial_P \hat{U}_2^M - \frac{1}{2}\eta^{MN}\eta_{PQ}\hat{U}_1^P \partial_N \hat{U}_2^Q \\ - \hat{U}_2^P \partial_P \hat{U}_1^M + \frac{1}{2}\eta^{MN}\eta_{PQ}\hat{U}_2^P \partial_N \hat{U}_1^Q\end{aligned} \tag{4.47}$$

When expanded out this is clearly the analogue of the Courant bracket for which vector fields and forms are treated identically:[20]

$$\begin{aligned}[\hat{U}_1, \hat{U}_2]_C = [U_1, U_2] + \mathcal{L}_{U_1}\chi_2 - \mathcal{L}_{U_2}\chi_1 - \frac{1}{2}\mathrm{d}(\iota_{U_1}\chi_2 - \iota_{U_2}\chi_1) \\ + [\chi_1, \chi_2] + \mathcal{L}_{\chi_1}U_2 - \mathcal{L}_{\chi_2}U_1 - \frac{1}{2}\tilde{\mathrm{d}}(\tilde{\iota}_{\chi_1}U_2 - \tilde{\iota}_{\chi_2}U_1)\end{aligned} \tag{4.48}$$

When demanding $\tilde{\partial} = 0$ all of the terms on the second line become zero and we see that $[\hat{U}_1, \hat{U}_2]_C \xrightarrow{\tilde{\partial}=0} [U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}}$. Solving the section condition reduces the C-bracket to the Courant bracket.

Just as with the ansatz for the generalised Lie derivative (4.23), in a theory with an arbitrary duality group the bracket can be written:[24]

$$\begin{aligned}[\![\hat{U}_1, \hat{U}_2]\!]^M = \hat{U}_1^P \partial_P \hat{U}_2^M - \frac{1}{2}Y^{MN}{}_{PQ}\hat{U}_1^P \partial_N \hat{U}_2^Q \\ - \hat{U}_2^P \partial_P \hat{U}_1^M + \frac{1}{2}Y^{MN}{}_{PQ}\hat{U}_2^P \partial_N \hat{U}_1^Q\end{aligned} \tag{4.49}$$

Where for DFT this "exceptional Courant bracket" is the C-bracket.

From the C-bracket's definition, in (4.47), it is trivial to note that it is antisymmetric, just as the Lie bracket is seen to be in (4.9). The properties of this generalised diffeomorphism algebra will now be examined; in particular testing its closure, and whether it possesses a Jacobi identity.

‡The symbol B was reused here to keep this section consistent with the literature. Aside from in this brief discussion of the automorphism, the B-field will always refer to the Kalb-Ramond field.

4.5.2. Closure

Given that the C-bracket appears to be a suitable choice for the generalised diffeomorphisms, one may then ask if the generalised Lie derivative equipped with such a C-bracket forms an algebra. In this section the closure of the generalised Lie derivative under the C-bracket will be examined; i.e, does the derivative possess a homomorphism property of the type seen in (4.10) Specifically, the following case will be tested:

$$[\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_V]A_M \stackrel{?}{=} \hat{\mathcal{L}}_{[U,V]_C} A_M \tag{4.50}$$

Where U, V, and A are $O(d,d)$ objects of the type defined in (4.20).

Beginning first by explicitly calculating the left-hand side:

$$[\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_V]A_M = \hat{\mathcal{L}}_U \hat{\mathcal{L}}_V A_M - \hat{\mathcal{L}}_V \hat{\mathcal{L}}_U A_M \tag{4.51}$$

After calculating the first term we will be able to attain the second term by simply transposing U and V. Computing the derivative along V in this first term yields:

$$\hat{\mathcal{L}}_U \hat{\mathcal{L}}_V A_M = \hat{\mathcal{L}}_U [V^P(\partial_P A_M) + (\partial_M V^P)A_P - (\partial^P V_M)A_P] \tag{4.52}$$

Computing the generalised Lie derivative along U of each of these terms respectively:

$$\hat{\mathcal{L}}_U[V^P(\partial_P A_M)] = U^Q\partial_Q(V^P\partial_P A_M) + (\partial_M U^Q)V^P\partial_P A_Q - (\partial^Q U_M)V^P\partial_P A_Q \tag{4.53}$$

$$\hat{\mathcal{L}}_U[(\partial_M V^P)A_P] = U^Q\partial_Q(A_P\partial_M V^P) + (\partial_M U^Q)A_P\partial_Q V^P - (\partial^Q U_M)A_P\partial_Q V^P \tag{4.54}$$

$$-\hat{\mathcal{L}}_U[(\partial^P V_M)A_P] = -U^Q\partial_Q(A_P\partial^P V_M) - (\partial_M U^Q)A_P\partial^P V_Q + (\partial^Q U_M)A_P\partial^P V_Q \tag{4.55}$$

Using these results we can rewrite (4.51) in a fully expanded form, which after cancelling like terms reads:

$$\begin{aligned}[\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_V]A_M = \Big\{ & U^Q(\partial_Q V^P)(\partial_P A_M) + U^Q A_P(\partial_Q\partial_M V^P) + A_P(\partial_M U^Q)(\partial_Q V^P) \\ & + V^Q A_P(\partial_Q\partial^P U_M) + A_P(\partial^Q U_M)(\partial^P V_Q) + A_P(\partial_M V^Q)(\partial^P U_Q)\Big\} \\ & - \{U \leftrightarrow V\} + (\Delta_{\text{LHS}})_M \end{aligned} \tag{4.56}$$

Where, for conciseness, $\{U \leftrightarrow V\}$ is used to denote that the same terms as in the first set of braces are repeated with U and V transposed. Two terms appear that would vanish under imposition of the strong constraint, these have been collected into the object Δ_{LHS}:

$$(\Delta_{\text{LHS}})_M \coloneqq A_P(\partial^Q V_M)(\partial_Q U^P) - A_P(\partial^Q U_M)(\partial_Q V^P) \tag{4.57}$$

Turning now to the right-hand side of (4.50), we have:

$$\hat{\mathcal{L}}_{[U,V]_C}A_M = ([U,V]_C)^P\partial_P A_M + A_P\partial_M([U,V]_C)^P - A_P\partial^P([U,V]_C)_M \tag{4.58}$$

After computing the C-Bracket, and expanding terms using the product rule of partial derivatives, the first term can be rewritten as:

$$([U,V]_C)^P\partial_P A_M = U^Q(\partial_Q V^P)(\partial_P A_M) - V^Q(\partial_Q U^P)(\partial_P A_M) + (\Delta_{\text{RHS}})_M \tag{4.59}$$

Where terms that vanish under the strong constraint have been collected into the object Δ_{RHS}:

$$(\Delta_{\text{RHS}})_M \coloneqq \frac{1}{2}V_Q(\partial^P U^Q)(\partial_P A_M) - \frac{1}{2}U_Q(\partial^P V^Q)(\partial_P A_M) \tag{4.60}$$

The second term in (4.58) can be rewritten:

$$\begin{aligned} A_P\partial_M([U,V]_C)^P = \Big\{ & A_P U^Q(\partial_M\partial_Q V^P) + A_P(\partial_M V^Q)(\partial_Q V^P) \\ & + \frac{1}{2}A_P V_Q(\partial_M\partial^P U^Q) + \frac{1}{2}A_P(\partial_M V_Q)(\partial^P V^Q)\Big\} \\ & - \{U \leftrightarrow V\} \end{aligned} \tag{4.61}$$

Similarly, the third term in (4.58) can be rewritten as:

$$\begin{aligned} -A_P\partial^P([U,V]_C)_M = \Big\{ & A_P V^Q(\partial^P\partial_Q U_M) + A_P(\partial^P U^Q)(\partial_Q V_M) \\ & - \frac{1}{2}A_P V_Q(\partial^P\partial_M U^Q) + \frac{1}{2}A_P(\partial_M V^Q)(\partial^P U_Q)\Big\} \\ & - \{U \leftrightarrow V\} \end{aligned} \tag{4.62}$$

These results can then be used to expand out (4.58), which after cancelling/collecting like terms becomes:

$$\begin{aligned} \hat{\mathcal{L}}_{[U,V]_C}A_M = \Big\{ & U^Q(\partial_Q V^P)(\partial_P A_M) + U^Q A_P(\partial_Q\partial_M V^P) + A_P(\partial_M U^Q)(\partial_Q V^P) \\ & + V^Q A_P(\partial_Q\partial^P U_M) + A_P(\partial^Q U_M)(\partial^P V_Q) + A_P(\partial_M V^Q)(\partial^P U_Q)\Big\} \\ & - \{U \leftrightarrow V\} + (\Delta_{\text{RHS}})_M \end{aligned} \tag{4.63}$$

Upon comparing (4.56) and (4.63) one finds that the left-hand side and right-hand side of (4.50) are identical up to strong constraint-like terms; i.e,

$$[\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_V]A_M = \hat{\mathcal{L}}_{[U,V]_C} A_M + \Delta_M \tag{4.64}$$

with $\Delta \coloneqq \Delta_{\text{LHS}} - \Delta_{\text{RHS}}$. This is an interesting result, and tells us that the generalised Lie derivative is only closed under the C-bracket when the strong constraint of DFT is imposed. This provides additional motivation for the strong constraint; without it the generalised diffeomorphisms do not obey a closure condition.

In general, upon imposing the strong constraint, the closure property is: [13]

$$[\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_V] = \hat{\mathcal{L}}_{[U,V]_C} \tag{4.65}$$

4.5.3. Jacobi Identity

The final property required for the C-bracket to form an algebra is the existence of a Jacobi identity of the form seen in (4.11). We can define a *Jacobiator*:

$$J(U, V, W) \coloneqq [\hat{\mathcal{L}}_U, [\hat{\mathcal{L}}_V, \hat{\mathcal{L}}_W]] + [\hat{\mathcal{L}}_V, [\hat{\mathcal{L}}_W, \hat{\mathcal{L}}_U]] + [\hat{\mathcal{L}}_W, [\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_V]] \tag{4.66}$$

Where U, V, and W are $O(d, d)$ objects defined in the same manner as (4.20). A Jacobi identity only exists if $J(U, V, W) = 0$ for arbitrary U, V, and W. After using the closure result from 4.5.2, with the strong constraint imposed, the Jacobiator can be rewritten as:

$$\begin{aligned} J(U, V, W) &= [\hat{\mathcal{L}}_U, \hat{\mathcal{L}}_{[V,W]_C}] + [\hat{\mathcal{L}}_V, \hat{\mathcal{L}}_{[W,U]_C}] + [\hat{\mathcal{L}}_W, \hat{\mathcal{L}}_{[U,V]_C}] && (4.67) \\ &= \hat{\mathcal{L}}_{[U,[V,W]_C]_C} + \hat{\mathcal{L}}_{[V,[W,U]_C]_C} + \hat{\mathcal{L}}_{[W,[U,V]_C]_C} && (4.68) \\ &= \hat{\mathcal{L}}_{[U,[V,W]_C]_C + [V,[W,U]_C]_C + [W,[U,V]_C]_C} && (4.69) \end{aligned}$$

Hence, it is useful to define a second Jacobiator:

$$K(U, V, W) \coloneqq [U, [V, W]_C]_C + [V, [W, U]_C]_C + [W, [U, V]_C]_C \tag{4.70}$$

Before we calculate these Jacobiators it will be favourable to calculate two useful identities. The first is the antisymmetric combination of generalised Lie derivatives:

$$\begin{aligned} \hat{\mathcal{L}}_U V^M - \hat{\mathcal{L}}_V U^M = U^P(\partial_P V^M) - (\partial_P U^M)V^P + (\partial^M U_P)V^P \\ - V^P(\partial_P U^M) + (\partial_P V^M)U^P - (\partial^M V_P)U^P \end{aligned} \tag{4.71}$$

After collecting like terms, the right-hand side can be identified with the C-bracket:

$$\hat{\mathcal{L}}_U V^M - \hat{\mathcal{L}}_V U^M = 2([U,V]_C)^M \tag{4.72}$$

The second identity relates to the symmetric combination of generalised Lie derivatives:

$$\begin{aligned}\hat{\mathcal{L}}_U V^M + \hat{\mathcal{L}}_V U^M = U^P(\partial_P V^M) - (\partial_P U^M)V^P + (\partial^M U_P)V^P \\ + V^P(\partial_P U^M) - (\partial_P V^M)U^P + (\partial^M V_P)U^P\end{aligned} \tag{4.73}$$

After cancelling like terms we are left with:

$$\hat{\mathcal{L}}_U V^M + \hat{\mathcal{L}}_V U^M = (\partial^M U_P)V^P + (\partial^M V_P)U^P \tag{4.74}$$

$$= \partial^M (U_P V^P) \tag{4.75}$$

With these identities in place, the Jacobiator K can be calculated. Using (4.72), K can be written as:

$$K(U,V,W)^M = \frac{1}{2}\hat{\mathcal{L}}_U([V,W]_C)^M - \frac{1}{2}\hat{\mathcal{L}}_{[V,W]_C} U^M + \mathit{cyclic} \tag{4.76}$$

Where, for conciseness, *cyclic* has been used to denote that all of the terms are repeated with U, V and W cyclically permuted, for example:

$$\hat{\mathcal{L}}_U([V,W]_C)^M + \mathit{cyclic} = \hat{\mathcal{L}}_U([V,W]_C)^M + \hat{\mathcal{L}}_V([W,U]_C)^M + \hat{\mathcal{L}}_V([W,U]_C)^M \tag{4.77}$$

By using the closure property from 4.5.2, the expression for K becomes:

$$K(U,V,W)^M = \frac{1}{2}\hat{\mathcal{L}}_U([V,W]_C)^M - \frac{1}{2}[\hat{\mathcal{L}}_V, \hat{\mathcal{L}}_W]U^M + \mathit{cyclic} \tag{4.78}$$

Expanding this out, again taking advantage of (4.72), yields:

$$\begin{aligned}K(U,V,W)^M = \frac{1}{4}\hat{\mathcal{L}}_U\hat{\mathcal{L}}_V W^M - \frac{1}{4}\hat{\mathcal{L}}_U\hat{\mathcal{L}}_W V^M \\ - \frac{1}{2}\hat{\mathcal{L}}_V\hat{\mathcal{L}}_W U^M + \frac{1}{2}\hat{\mathcal{L}}_W\hat{\mathcal{L}}_V U^M + \mathit{cyclic}\end{aligned} \tag{4.79}$$

Explicitly writing out the terms inside *cyclic* gives:

$$
\begin{aligned}
K(U,V,W)^M = \frac{1}{4}\Big(&\hat{\mathcal{L}}_U\hat{\mathcal{L}}_V W^M + \hat{\mathcal{L}}_V\hat{\mathcal{L}}_W U^M + \hat{\mathcal{L}}_W\hat{\mathcal{L}}_U V^M \\
&- \hat{\mathcal{L}}_U\hat{\mathcal{L}}_W V^M - \hat{\mathcal{L}}_V\hat{\mathcal{L}}_U W^M - \hat{\mathcal{L}}_W\hat{\mathcal{L}}_V U^M\Big) \\
+ \frac{1}{2}\Big(&\hat{\mathcal{L}}_W\hat{\mathcal{L}}_V U^M + \hat{\mathcal{L}}_U\hat{\mathcal{L}}_W V^M + \hat{\mathcal{L}}_V\hat{\mathcal{L}}_U W^M \\
&- \hat{\mathcal{L}}_V\hat{\mathcal{L}}_W U^M - \hat{\mathcal{L}}_W\hat{\mathcal{L}}_U V^M - \hat{\mathcal{L}}_U\hat{\mathcal{L}}_V W^M\Big)
\end{aligned}
\tag{4.80}
$$

Each of the terms in the first set of parentheses can be pair with a term in the second. Collecting these terms together yields:

$$
\begin{aligned}
K(U,V,W)^M = \frac{1}{4}\Big(&\hat{\mathcal{L}}_W\hat{\mathcal{L}}_V U^M + \hat{\mathcal{L}}_U\hat{\mathcal{L}}_W V^M + \hat{\mathcal{L}}_V\hat{\mathcal{L}}_U W^M \\
&- \hat{\mathcal{L}}_V\hat{\mathcal{L}}_W U^M - \hat{\mathcal{L}}_W\hat{\mathcal{L}}_U V^M - \hat{\mathcal{L}}_U\hat{\mathcal{L}}_V W^M\Big)
\end{aligned}
\tag{4.81}
$$

One way of simplifying this expression is to use again the identity (4.72):

$$K(U,V,W)^M = \frac{1}{2}\hat{\mathcal{L}}_W([V,U]_C)^M + \frac{1}{2}\hat{\mathcal{L}}_V([U,W]_C)^M + \frac{1}{2}\hat{\mathcal{L}}_U([W,V]_C)^M \tag{4.82}$$

$$= -\frac{1}{2}\hat{\mathcal{L}}_U([V,W]_C)^M + cyclic \tag{4.83}$$

Alternatively, (4.81) can be simplified by utilising the closure property from 4.5.2:

$$K(U,V,W)^M = \frac{1}{4}\hat{\mathcal{L}}_{[W,V]_C}U^M + \frac{1}{4}\hat{\mathcal{L}}_{[U,W]_C}V^M + \frac{1}{4}\hat{\mathcal{L}}_{[V,U]_C}W^M \tag{4.84}$$

$$= -\frac{1}{4}\hat{\mathcal{L}}_{[V,W]_C}U^M + cyclic \tag{4.85}$$

Equations (4.83) and (4.85) can then be summed to give:

$$2K(U,V,W)^M = -\frac{1}{2}\hat{\mathcal{L}}_U([V,W]_C)^M - \frac{1}{4}\hat{\mathcal{L}}_{[V,W]_C}U^M + cyclic \tag{4.86}$$

The identity (4.75) can then be used to introduce a partial derivative term:

$$2K(U,V,W)^M = -\frac{1}{4}\hat{\mathcal{L}}_U([V,W]_C)^M - \frac{1}{4}\partial^M\Big(U_P([V,W]_C)^P\Big) + cyclic \tag{4.87}$$

Using (4.83) this becomes:

$$2K(U,V,W)^M = \frac{1}{2}K(U,V,W)^M - \frac{1}{4}\partial^M\Big(U_P([V,W]_C)^P\Big) + cyclic \tag{4.88}$$

Finally, collecting the K terms gives the Jacobiator as:

$$K(U,V,W)^M = -\frac{1}{6}\partial^M\Big(U_P([V,W]_C)^P + cyclic\Big) \tag{4.89}$$

This Jacobiator for the C-bracket is non-zero. Hence, the C-bracket does not obey a Jacobi identity and does not form a Lie algebra. Instead it can be said to form an algebroid. If one were to apply the second condition to this equation they would attain a Jacobiator for the Courant bracket, which is similarly non-zero. Thus the previous claim that the Courant bracket forms a Courant algebroid, not a Lie algebra, is justified.

Whilst non-zero, the Jacobiator for the C-bracket is clearly a total derivative. The infinitesimal transformations generated by total derivatives are trivial, and, as such, the generalised Lie derivative along K is always zero. Hence, we have the Jacobiator of the generalised Lie derivative:

$$J(U,V,W) = \hat{\mathcal{L}}_{K(U,V,W)} = 0 \tag{4.90}$$

Thus, whilst the C-bracket does not possess a Jacobi identity, the generalised Lie derivative equipped with the C-bracket does. This is good news for our generalised diffeomorphisms. The generalised Lie derivative will be sufficient to describe the local symmetries of DFT.

5. An Action for DFT

5.1. The DFT Action

In this chapter we will consider the following DFT action:[14]

$$S_{\text{DFT}} = \int \mathrm{d}x\,\mathrm{d}\tilde{x}\; e^{-2d}\bigg(\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_L\mathcal{H}_{KN} \\ - 2\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d\,\partial_N d\bigg) \tag{5.1}$$

Such an action is constructed from $O(d,d)$ scalars. Thus, by the same reasoning as discussed in section 3.2, it is manifestly invariant under $O(d,d)$ transformations. This action is also invariant under the local transformations generated by the generalised diffeomorphisms of DFT.[25] If the action presented here is a valid description of DFT, we expect to be able to recover the original d-dimensional theory by imposing the section condition. This will be tested in the next section, whereby the strong constraint is solved by choosing $\tilde{\partial} = 0$.

5.2. Reduction of DFT to the Supergravity Action

It will now be shown that in the limit $\tilde{\partial} = 0$ the DFT action, (5.1), reduces to the standard NS-NS sector supergravity action; i.e.,[11]

$$S_{\text{DFT}} \xrightarrow{\tilde{\partial}=0} S_{\text{SG}} = \int \mathrm{d}x\,\sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}H^2\right) \tag{5.2}$$

Where R is the Ricci scalar, Φ is the dilaton, and H is the NS sector three-form field strength. Φ and H written in terms of d and B are:

$$\sqrt{|G|}e^{2\Phi} = e^{2d} \qquad H \coloneqq \mathrm{d}B \tag{5.3, 5.4}$$

This expression for the dilaton, (5.3), follows directly from the DFT dilaton's definition in equation (3.14).

Demanding this independence from the dual coordinates yields the following partial derivatives:

$$\partial_M \xrightarrow{\tilde{\partial}=0} \begin{pmatrix} 0 \\ \partial_i \end{pmatrix} \qquad \partial^M \xrightarrow{\tilde{\partial}=0} \begin{pmatrix} \partial_i \\ 0 \end{pmatrix} \tag{5.5, 5.6}$$

For conciseness, the first two terms in (5.1) will be written as follows:

$$\mathscr{L}_1 := \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL}\Big|_{\tilde{\partial}=0} \tag{5.7}$$

$$\mathscr{L}_2 := -\frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_L\mathcal{H}_{KN}\Big|_{\tilde{\partial}=0} \tag{5.8}$$

Evaluating the $\tilde{\partial} = 0$ limit in (5.7) yields

$$\mathscr{L}_1 = \frac{1}{8}\mathcal{H}^{ij}\partial_i\mathcal{H}^{KL}\partial_j\mathcal{H}_{KL} \tag{5.9}$$

Where here the labels i and j only run over the coordinates for which derivatives in (5.5) are non-zero, i.e., $i, j = D+1, ..., 2D$. This limit effectively selects specific elements from the generalised metric, thus:

$$\begin{aligned} \mathscr{L}_1 = \frac{1}{8}G^{ij}\Big[&\partial_i(G - BG^{-1}B)_{kl}\partial_j G^{kl} + \partial_i(BG)_k{}^l\partial_j(-GB)^k{}_l \\ &+ \partial_i(-GB)^k{}_l\partial_j(BG)_k{}^l + \partial_i G^{kl}\partial_j(G - BG^{-1}B)_{kl}\Big] \end{aligned} \tag{5.10}$$

These lowercase indices now run the entire span of G and B, i.e., from 0 to D. Collecting like terms we then see that:

$$\mathscr{L}_1 = \frac{1}{4}G^{ij}\Big[\partial_i G_{kl}\partial_j G^{kl} - \partial_i(B_{kp}G^{pl})\partial_j(G^{kq}B_{ql}) - \partial_i(B_{kp}G^{pq}B_{ql})\partial_j G^{kl}\Big] \tag{5.11}$$

The proposal then is to decouple derivatives of the metric, G, from derivatives of the Kalb-Ramond field, B. The expectation then is that the metric-only terms will form part of the scalar curvature, and the B-field terms will correspond to the NS field strength H.

After applying Leibniz's product rule several times, one sees that $\mathscr{L}_1$ can be simplified to the following form:

$$\mathscr{L}_1 = \frac{1}{4}G^{ij}\partial_i G_{kl}\partial_j G^{kl} - \frac{1}{4}G^{ij}G^{pq}G^{kl}\partial_i B_{ql}\partial_j B_{pk} \tag{5.12}$$

Thus derivatives of G and B have been decoupled as planned.

Turning now to $\mathscr{L}_2$, we see that evaluating the limit in (5.8) yields:

$$\mathscr{L}_2 = -\frac{1}{2}\mathcal{H}^{iN}\partial_i\mathcal{H}^{Kj}\partial_j\mathcal{H}_{KN} \tag{5.13}$$

Which expanded in terms of G and B fields is:

$$\begin{aligned}\mathscr{L}_2 = &-\frac{1}{2}G^{in}\Big[-\partial_i(BG)_k{}^j\partial_j(GB)^k{}_n + \partial_i G^{kj}\partial_j(G - BG^{-1}B)_{kn}\Big]\\ &+\frac{1}{2}(GB)^i{}_n\Big[\partial_i(BG)_k{}^j\partial_j G^{kn} + \partial_i G^{kj}\partial_j(BG)_k{}^n\Big]\end{aligned} \tag{5.14}$$

As with $\mathscr{L}_1$, the derivatives of the G and B fields can now be decoupled. With some work one attains:

$$\mathscr{L}_2 = -\frac{1}{2}G^{in}\partial_i G^{kj}\partial_j G_{kn} - \frac{1}{2}G^{in}G^{kj}G^{pq}\partial_i B_{kp}\partial_j B_{qn} \tag{5.15}$$

Now all of the B field terms have been attained we can identify them with the H^2 term in (5.2). First let's consider the definitions of B and H as a two-form and a three-form respectively:

$$B \coloneqq \frac{1}{2!}B_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \tag{5.16}$$

$$H \coloneqq \frac{1}{3!}H_{ijk}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \tag{5.17}$$

Since (5.4) states that H is the exterior derivative of B we then see that:

$$H_{ijk} = 3\partial_{[i}B_{jk]} \tag{5.18}$$

$$= \frac{1}{2}\sum_{\pi\in\mathcal{S}_3}\mathrm{sgn}(\pi)\partial_{\pi(i)}B_{\pi(j)\pi(k)} \tag{5.19}$$

Where $\mathcal{S}_3$ is the group of all permutations of a set of three objects, and $\mathrm{sgn}(\pi)$ gives the sign of each permutation. For example, when $\pi = (1\ 2\ 3)$ we have $\pi(i) = j$, $\pi(j) = k$, and $\pi(k) = i$. Since B is antisymmetric we can rewrite H_{ijk} as:

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \tag{5.20}$$

Using this relation, we can now rewrite the H^2 term in (5.2):

$$-\frac{1}{12}H^2 = -\frac{1}{12}G^{ip}G^{jq}G^{kr}H_{ijk}H_{pqr} \tag{5.21}$$

$$= -\frac{1}{4}G^{ip}G^{jq}G^{kr}\partial_i B_{jk}(\partial_p B_{qr} + \partial_q B_{rp} + \partial_r B_{pq}) \tag{5.22}$$

After appropriately relabelling summation indices it can be seen the final two terms in (5.22) are equivalent. Hence:

$$-\frac{1}{12}H^2 = -\frac{1}{4}G^{ip}G^{jq}G^{kr}\partial_i B_{jk}\partial_p B_{qr} - \frac{1}{2}G^{ip}G^{jq}G^{kr}\partial_i B_{jk}\partial_q B_{rp} \tag{5.23}$$

$$= -\frac{1}{4}G^{ij}G^{pq}G^{kl}\partial_i B_{ql}\partial_j B_{pk} - \frac{1}{2}G^{in}G^{kj}G^{pq}\partial_i B_{kp}\partial_j B_{qn} \tag{5.24}$$

Where both of these equations are identical up to a relabelling of indices. Thus the B field terms from (5.12) and (5.15) have been identified with the H^2 term in (5.2).

Now we will focus our attention to the two terms with derivatives of the dilaton:

$$S_d := \int \mathrm{d}x\,\mathrm{d}\tilde{x}\; e^{-2d}\Big(-2\partial_M d\;\partial_N \mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d\;\partial_N d\Big)\Big|_{\tilde{\partial}=0} \tag{5.25}$$

By evaluating this limit, and using (5.3) to write the result in terms of G and the NS-NS sector dilaton Φ, yields:

$$\begin{aligned} S_d = \int \mathrm{d}x\; \sqrt{|G|}e^{-2\Phi}\Big(&+G^{ij}\partial_i \ln\sqrt{|G|}\;\partial_j \ln\sqrt{|G|} + \partial_i \ln\sqrt{|G|}\;\partial_j G^{ij} \\ &- 2\partial_i\Phi\partial_j G^{ij} - 4G^{ij}\partial_i\Phi\;\partial_j\ln\sqrt{|G|} + 4(\partial\Phi)^2\Big) \end{aligned} \tag{5.26}$$

Using the chain rule for differentiation, along with Jacobi's formula for the derivative of a matrix determinant, yields the following:

$$\partial_i \ln\sqrt{|G|} = \frac{1}{2}|G^{-1}|\partial_i|G| \tag{5.27}$$

$$= \frac{1}{2}\mathrm{tr}[G^{-1}\partial_i G] \tag{5.28}$$

This is none other than the Christoffel symbol with two indices contracted, ${\Gamma^k}_{ik}$.

At this point it is advantageous to consider the DFT action again as a whole:

$$S_{\text{DFT}} \xrightarrow{\tilde{\partial}=0} \int \mathrm{d}x \ \sqrt{|G|}e^{-2\Phi}\bigg(\frac{1}{4}G^{ij}\partial_i G_{kl}\partial_j G^{kl} - \frac{1}{2}G^{in}\partial_i G^{kj}\partial_j G_{kn} \\ + G^{ij}\Gamma^k{}_{ik}\Gamma^m{}_{jm} + \Gamma^k{}_{ik}\partial_j G^{ij} - 2\partial_i\Phi\partial_j G^{ij} \\ - 4G^{ij}\Gamma^k{}_{ik}\partial_j\Phi + 4(\partial\Phi)^2 - \frac{1}{12}H^2\bigg) \tag{5.29}$$

All that remains now is to identify the terms containing derivatives of the metric with the Ricci scalar in the supergravity action (5.2). We can write the classical Ricci scalar:

$$\int \mathrm{d}x \ e^{-2\Phi}R = \int \mathrm{d}x \ e^{-2\Phi}G^{ij}\Big(\partial_k\Gamma^k{}_{ij} - \partial_j\Gamma^k{}_{ik} + \Gamma^m{}_{ij}\Gamma^k{}_{mk} - \Gamma^m{}_{ik}\Gamma^k{}_{jm}\Big) \tag{5.30}$$

Each Christoffel symbol contains first derivatives of the metric; hence, this expression contains second derivatives of the metric. Our DFT action only contains first derivatives of the metric. It is necessary to integrate by parts any derivatives of Christoffel symbols, discarding boundary terms as we do so, namely:

$$\int \mathrm{d}x \ e^{-2\Phi}G^{ij}\partial_k\Gamma^k{}_{ij} = -\int \mathrm{d}x \ \Gamma^k{}_{ij}\partial_k(G^{ij}e^{-2\Phi}) \tag{5.31}$$

$$= \int \mathrm{d}x \ e^{-2\Phi}\Gamma^k{}_{ij}(2G^{ij}\partial_k\Phi - \partial_k G^{ij}) \tag{5.32}$$

$$-\int \mathrm{d}x \ e^{-2\Phi}G^{ij}\partial_j\Gamma^k{}_{ik} = \int \mathrm{d}x \ e^{-2\Phi}\Gamma^k{}_{ik}(\partial_j G^{ij} - 2G^{ij}\partial_j\Phi) \tag{5.33}$$

With some work one can see that the first term in (5.32) can be reexpressed:

$$2G^{ij}\Gamma^k{}_{ij}\partial_k\Phi = -2\partial_i\Phi\partial_j G^{ij} - 2G^{ij}\Gamma^k{}_{ik}\partial_j\Phi \tag{5.34}$$

Using this result, in conjunction with (5.32) and (5.33), the Ricci scalar can be rewritten:

$$\int \mathrm{d}x \ e^{-2\Phi}R = \int \mathrm{d}x \ e^{-2\Phi}G^{ij}\Big(\Gamma^m{}_{ij}\Gamma^k{}_{mk} - \Gamma^m{}_{ik}\Gamma^k{}_{jm} - \Gamma^k{}_{ij}\partial_k G^{ij} \\ + \Gamma^k{}_{ik}\partial_j G^{ij} - 2\partial_i\Phi\partial_j G^{ij} - 4G^{ij}\Gamma^k{}_{ik}\partial_j\Phi\Big) \tag{5.35}$$

With some work one can reexpress the above form of the Ricci scalar as:

$$\int \mathrm{d}x\ e^{-2\Phi}R = \int \mathrm{d}x\ e^{-2\Phi}G^{ij}\Big(\frac{1}{4}G^{ij}\partial_i G_{kl}\partial_j G^{kl} - \frac{1}{2}G^{in}\partial_i G^{kj}\partial_j G_{kn} + G^{ij}\Gamma^k{}_{ik}\Gamma^m{}_{jm} + \Gamma^k{}_{ik}\partial_j G^{ij} - 2\partial_i\Phi\partial_j G^{ij} - 4G^{ij}\Gamma^k{}_{ik}\partial_j\Phi\Big) \tag{5.36}$$

Upon identifying this expression with terms in (5.29) one obtains the following:

$$S_{\mathrm{DFT}} \xrightarrow{\tilde{\partial}=0} \int \mathrm{d}x\ \sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}H^2\right) \tag{5.37}$$

This is identically the supergravity action shown in (5.2). Thus it has been shown that, by demanding that all dependence on the dual coordinates vanishes, the DFT action does indeed reduce to the standard NS-NS sector supergravity action.

The reduction of this DFT action back to the NS-NS sector supergravity action is an important result. It shows that the action presented in this chapter is indeed valid action for DFT. It is gauge invariant, manifestly $O(d,d)$ invariant, and upon imposition of the section condition it does indeed reduce back to the standard d-dimensional theory.

6. Conclusion

This report began with a discussion of the T-duality of closed bosonic string theories in the presence of isometries. It was then seen that these hidden symmetries are in fact described by the action of $O(d,d,\mathbb{Z})$ matrices on the generalised metric. Following this a Double Field Theory was constructed. By doubling the dimensions of the background space-time, it became possible to write a theory manifestly invariant under the continuous $O(d,d,\mathbb{R})$ group. In doing so, the T-duality transformations were promoted to manifest symmetries. The T-dual theories are then special cases of the DFT, found by solving the section condition in backgrounds with isometries. One could say that the DFT is the unification of the T-dual theories.

Constructing the DFT also allowed the diffeomorphisms and two-form gauge transformations of the d-dimensional theory to be combined into generalised diffeomorphisms on a $2d$-dimensional space-time. This is reminiscent of the approach taken in Kaluza-Klein theories to unify diffeomorphisms with U(1) gauge transformations. These local symmetries of DFT are described by the generalised Lie derivative, equipped with the C-bracket. Such a description required the strong constraint to be included in the theory, without which the generalised Lie derivative does not close and does not obey a Jacobi identity. However, even with the strong constraint imposed, the C-bracket has a non-zero Jacobiator and so does not form a Lie algebra. Instead it forms an algebroid. Following this discussion of local symmetries, a manifestly $O(d,d)$ invariant action was presented. It was shown that, upon imposition of the section condition, this action on the doubled space-time reduced to the d-dimensional NS-NS sector supergravity action.

Given this DFT approach to T-duality, one may ask if the other dualities between string theories can be promoted to manifest symmetries in a similar manner. Exceptional Field Theories (EFTs) are constructed in a similar way to DFT, with different duality groups taking the place of the $O(d,d,\mathbb{R})$ group. U-duality combines the S-dualities and T-dualities through which all sectors of string theory are related. It would be desirable then to construct an EFT in which U-dualities are promoted to manifest symmetries. All of the sectors of string theory would then be special cases of this EFT. In a sense, such a theory would unify all of the different string theory approaches.

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A. The Lie Derivative of an Arbitrary Tensor Field

In this appendix the Lie derivative of a rank (n, m) tensor field, $T^{\mu_1 \ldots \mu_n}_{\sigma_1 \ldots \sigma_m}$, along some vector field U, will be examined. From the discussion in section 4.1, it follows that to compute such a Lie derivative one would need to pushforward the n contravariant indices and pullback the m covariant indices. This process is simplified by realising that for the group of diffeomorphisms the pullback is just the inverse of the pushforward. In analogy with the Lie derivative of a vector field defined in equation (4.5), the Lie derivative of this mixed tensor field is given by:

$$\mathcal{L}_U T^{\mu_1 \ldots \mu_n}_{\sigma_1 \ldots \sigma_m} = \lim_{\xi \to 0} \frac{1}{\xi} \Bigg\{ \left(\frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\nu_n}} \right) \left(\frac{\partial x^{\rho_1}}{\partial x'^{\sigma_1}} \cdots \frac{\partial x^{\rho_m}}{\partial x'^{\sigma_m}} \right) T^{\nu_1 \ldots \nu_n}_{\rho_1 \ldots \rho_m}(x + \xi U) - T^{\mu_1 \ldots \mu_n}_{\sigma_1 \ldots \sigma_m}(x) \Bigg\} \tag{A.1}$$

Where the coordinates x'^i in the pushforward/pullback transformations are given by:

$$x'^\alpha = x^\alpha - \xi U^\alpha \tag{A.2}$$

Each of the contravariant transformations is then:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu - \xi \frac{\partial U^\mu}{\partial x^\nu} \tag{A.3}$$

The covariant transformations are:

$$\frac{\partial x^\rho}{\partial x'^\sigma} = \delta^\rho{}_\sigma + \xi \frac{\partial U^\rho}{\partial x'^\sigma} \tag{A.4}$$

$$= \delta^\rho{}_\sigma + \xi \frac{\partial U^\rho}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x'^\sigma} \tag{A.5}$$

$$= \delta^\rho{}_\sigma + \xi \frac{\partial U^\rho}{\partial x^\gamma} \left(\delta^\gamma{}_\sigma + \xi \frac{\partial U^\gamma}{\partial x'^\sigma} \right) \tag{A.6}$$

$$= \delta^\rho{}_\sigma + \xi \frac{\partial U^\rho}{\partial x^\sigma} + O(\xi^2) \tag{A.7}$$

Since we are taking the limit $\xi \to 0$ we will treat any terms of order ξ^2 or higher as negligible.

The tensor field $T^{\nu_1 \dots \nu_n}_{\rho_1 \dots \rho_m}(x^i + \xi U^i)$ can be Taylor expanded to give:

$$T^{\nu_1 \dots \nu_n}_{\rho_1 \dots \rho_m}(x^\alpha + \xi U^\alpha) = T^{\nu_1 \dots \nu_n}_{\rho_1 \dots \rho_m}(x) + \xi U^\alpha \frac{T^{\nu_1 \dots \nu_n}_{\rho_1 \dots \rho_m}(x)}{\partial x^\alpha} + O(\xi^2) \tag{A.8}$$

This result, along with the results from (A.2) and (A.3), can then be substituted back into equation (A.1):

$$\begin{aligned}\mathcal{L}_U T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m} = \lim_{\xi \to 0} \frac{1}{\xi} \Bigg\{ \left(\delta^{\mu_1}_{\ \nu_1} - \xi \frac{\partial U^{\mu_1}}{\partial x^{\nu_1}} \right) \dots \left(\delta^{\mu_n}_{\ \nu_n} - \xi \frac{\partial U^{\mu_n}}{\partial x^{\nu_n}} \right) \\ \times \left(\delta^{\rho_1}_{\ \sigma_1} + \xi \frac{\partial U^{\rho_1}}{\partial x^{\sigma_1}} \right) \dots \left(\delta^{\rho_m}_{\ \sigma_m} + \xi \frac{\partial U^{\rho_m}}{\partial x^{\sigma_m}} \right) \\ \times \left(T^{\nu_1 \dots \nu_n}_{\rho_1 \dots \rho_m} + \xi U^\alpha \frac{T^{\nu_1 \dots \nu_n}_{\rho_1 \dots \rho_m}}{\partial x^\alpha} \right) - T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m} \Bigg\}\end{aligned} \tag{A.9}$$

This expression is then simplified greatly by only keeping terms to first-order in ξ:

$$\begin{aligned}\mathcal{L}_U T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m} = \lim_{\xi \to 0} \frac{1}{\xi} \Bigg\{ T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m} + \xi U^\alpha (\partial_\alpha T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m}) + \xi \sum_{i=1}^{m} (\partial_{\sigma_i} U^{\rho_i}) T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \rho_i \dots \sigma_m} \\ - \xi \sum_{j=1}^{n} (\partial_{\nu_j} U^{\mu_j}) T^{\mu_1 \dots \nu_j \dots \mu_n}_{\sigma_1 \dots \sigma_m} - T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m} \Bigg\}\end{aligned} \tag{A.10}$$

Hence we arrive at the following expression for the Lie derivative of an arbitrary rank (n, m) tensor field:

$$\mathcal{L}_U T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m} = U^\rho (\partial_\rho T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \sigma_m}) + \sum_{i=1}^{m} (\partial_{\sigma_i} U^\rho) T^{\mu_1 \dots \mu_n}_{\sigma_1 \dots \rho \dots \sigma_m} - \sum_{j=1}^{n} (\partial_\rho U^{\mu_j}) T^{\mu_1 \dots \rho \dots \mu_n}_{\sigma_1 \dots \sigma_m} \tag{A.11}$$

B. The Automorphism of the Courant Bracket

In this appendix it will be verified that the B-field transformations are automorphisms of the Courant bracket. Such a B-field transformation takes the form:

$$[U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} \to [U_1 + \chi_1 + \iota_{U_1} B,\, U_2 + \chi_2 + \iota_{U_2} B]_{\text{Cour}} \tag{B.1}$$

Where B is any closed two-form. Such a B-transformed Courant bracket can then be expanded using the bracket's definition, (4.44), yielding:

$$\begin{aligned}[U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} \to{}& [U_1, U_2] + \mathcal{L}_{U_1}(\chi_2 + \iota_{U_2} B) - \mathcal{L}_{U_2}(\chi_1 + \iota_{U_1} B) \\ & - \frac{1}{2}\,\mathrm{d}[\iota_{U_1}(\chi_2 + \iota_{U_2} B) - \iota_{U_2}(\chi_1 + \iota_{U_1} B)]\end{aligned} \tag{B.2}$$

This can than be rewritten as:

$$\begin{aligned}[U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} \to{}& [U_1, U_2] + \mathcal{L}_{U_1}\chi_2 + \mathcal{L}_{U_1}\iota_{U_2} B - \mathcal{L}_{U_2}\chi_1 - \mathcal{L}_{U_2}\iota_{U_1} B \\ & - \frac{1}{2}\,\mathrm{d}[\iota_{U_1}\chi_2 + \iota_{U_1}\iota_{U_2} B - \iota_{U_2}\chi_1 + \iota_{U_2}\iota_{U_1} B]\end{aligned} \tag{B.3}$$

By identifying terms with the untransformed Courant bracket we have:

$$\begin{aligned}[U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} \to{}& [U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} + \mathcal{L}_{U_1}\iota_{U_2} B - \mathcal{L}_{U_2}\iota_{U_1} B \\ & - \frac{1}{2}\,\mathrm{d}[\iota_{U_1}\iota_{U_2} B + \iota_{U_2}\iota_{U_1} B]\end{aligned} \tag{B.4}$$

The final term can then be rewritten using the anticommutator of d and ι:

$$\mathrm{d}[\iota_{U_1}\iota_{U_2} B + \iota_{U_2}\iota_{U_1} B] = \{\mathrm{d}, \iota_{U_1}\}\iota_{U_2} B - \iota_{U_1}\,\mathrm{d}\iota_{U_2} B - \{\mathrm{d}, \iota_{U_2}\}\iota_{U_1} B + \iota_{U_2}\,\mathrm{d}\iota_{U_1} B \tag{B.5}$$

Cartan's identity, $\mathcal{L}_U = \{\mathrm{d}, \iota_U\}$, can then be used to simplify this further:

$$\mathrm{d}[\iota_{U_1}\iota_{U_2} B + \iota_{U_2}\iota_{U_1} B] = \mathcal{L}_{U_1}\iota_{U_2} B - \iota_{U_1}\,\mathrm{d}\iota_{U_2} B - \mathcal{L}_{U_2}\iota_{U_1} B + \iota_{U_2}\,\mathrm{d}\iota_{U_1} B \tag{B.6}$$

This same trick of introducing an anticommutator can be used again. After noting that $\mathrm{d}B = 0$ the above expression becomes:

$$\mathrm{d}[\iota_{U_1}\iota_{U_2}B + \iota_{U_2}\iota_{U_1}B] = \mathcal{L}_{U_1}\iota_{U_2}B - \iota_{U_1}\mathcal{L}_{U_2}B - \mathcal{L}_{U_2}\iota_{U_1}B + \iota_{U_2}\mathcal{L}_{U_1}B \tag{B.7}$$

Substituting this result back into equation (B.4) gives:

$$\begin{aligned}[U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} \rightarrow\ & [U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} + \frac{1}{2}(\mathcal{L}_{U_1}\iota_{U_2}B - \mathcal{L}_{U_2}\iota_{U_1}B \\ & + \iota_{U_1}\mathcal{L}_{U_2}B - \iota_{U_2}\mathcal{L}_{U_1}B)\end{aligned} \tag{B.8}$$

Finally, by using the identity $\iota_{[U_1,U_2]} = -\iota_{[U_2,U_1]} = [\mathcal{L}_{U_1}, \iota_{U_2}]$, one finds that:

$$[U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} \rightarrow\ [U_1 + \chi_1, U_2 + \chi_2]_{\text{Cour}} + \iota_{[U_1,U_2]}B \tag{B.9}$$

Thus, equation (4.46) has been verified and the B-field transformations have been shown to be automorphisms of the Courant bracket.

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